MATHEMATICS EXPLORATIONS

Detective-style Activities for the Real World

Aligns to the to the National Council of Teachers
of Mathematics Standards and to the
Common Core State Standards

David B. Spangler

GOOD YEAR BOOKS

V 2.0

Mathematics Explorations: Detective-style Activities for the Real World develops and reinforces key skills and concepts from each of the five Content Standards identified by the *Principles and Standards of School Mathematics* (National Council of Teachers of Mathematics). This book also addresses key skills and concepts from the Common Core State Standards for Mathematics. These alignments are shown with the Teacher's Notes for each activity lesson. *Mathematics Explorations* also fully implements the NCTM Process Standards and is aligned with the NCTM Curriculum Focal Points. See the Preface for more details.

Note: A separate Student Workbook is available. It includes all of the student activity lessons.

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Preface

It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment.
—Karl Friedrich Gauss, German mathematician (1777–1855)

In his landmark text, *The Process of Education,* Jerome Bruner, a founder of cognitive psychology, wrote, "Ideally, interest in the material to be learned is the best stimulus to learn" (1960, p. 12). According to the authors of *Helping Children Learn Mathematics,* "Success in mathematics learning requires being positively disposed toward the subject. Engaging oneself with mathematics requires frequent opportunities to make sense of it, to experience the rewards of making sense of it, and to recognize the benefits of perseverance" (2002, p. 16). The contexts of "mystery and exploration" that permeate the activity lessons in *Mathematics Explorations* are designed to stimulate and promote positive dispositions, beliefs, and attitudes related to the study of mathematics in general and to problem solving in particular. Students who delve into the challenging problems in this sourcebook will likely acquire and refine the disposition of perseverance—and should come away with the belief that exerting effort to learn mathematics is a worthwhile endeavor.

Mathematics Explorations is a sourcebook of engaging mathematics activity lessons that connect to the real world of students in grades 6–9. Both classroom teachers and students in mathematics methods courses should also benefit from the engaging, ready-to-use activity lessons and the professional development that is provided with the Teacher's Notes.

About This Book

Each activity lesson calls for students "to put on a detective hat" and search for patterns to discover important mathematical concepts and formulas, break a code, solve a mystery, conduct detective-type investigations, uncover and correct errors and blunders, analyze why a "trick" works, use clues to solve problems, and more. The activity lessons are designed to arouse student curiosity—resulting in their use of both inductive and deductive reasoning, critical thinking, graphical analysis, and other analytical skills needed for success in school and beyond in the world of work.

Each activity lesson in *Mathematics Explorations* is preceded by one or more pages of Teacher's Notes. The Teacher's Notes include a correlation to the *Common Core State Standards* (Council of Chief State School Officers and the National Governors Association Center for Best Practices, 2010) and to the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000). Also included is a list of the mathematics topics covered in the activity lesson and suggestions for student grouping. An extensive Background section provides deep mathematical background, historical notes, possible student responses, related Web sites, and other useful tips for teaching the activity lesson. A variety of mathematical quotations are included to high-light the fact that people from all walks of life have expressed a strong utility in the study of mathematics. The Teacher's Notes also include Mathematical Humor for each activity lesson. Full Solutions are included for all problems, and an Extension (with solutions) is provided for each activity lesson.

Although students may work through the activities independently, most are enhanced when students work collaboratively in pairs or in small groups. When collaboration occurs, students are encouraged to share and ponder ideas, listen to other points of view and solution processes, and divide the tasks necessary to complete the activity.

Correlation to the Common Core State Standards and to the NCTM Standards

In 2010, the National Governors Board and the Council of Chief State School Officers released the Common Core State Standards for Mathematics. These Standards are organized into two related categories: (1) Standards for Mathematical Content (defining what students should understand and be able to do) and (2) Standards for Mathematical Practice (describing ways in which students should engage with mathematics based on processes and proficiencies). The Standards may be downloaded at http://www.corestandards.org/the-standards. The activity lessons in this book align closely to both categories of standards.

In its *Principles and Standards for School Mathematics,* the National Council of Teachers of Mathematics (NCTM, 2000) addresses five Content Standards and five Process Standards (for acquiring and using the content knowledge) for grades preK–12. These Standards, listed below, are anchored on extensive foundational research on what works in the mathematics classroom.

NCTM Content Standards
Number and Operations Algebra Geometry Measurement Data Analysis and Probability
NCTM Process Standards
Problem Solving Reasoning and Proof Communication Connections Representation

Mathematics Explorations addresses key mathematical skills and concepts in each of the five NCTM Content Standards. The book is organized into chapters that provide a focus on each Content Standard. Because most of the activity lessons address multiple Content Standards, students experience how the various mathematics strands are connected. This integration of Content Standards is important for students in their quest to develop mathematical power. According to NCTM's *Principles and Standards for School Mathematics* (p. 64), "The notion that mathematical ideas are connected should permeate the school mathematics experience at all levels."

Mathematics Explorations also fully implements the essence of the NCTM Process Standards. This sourcebook delivers standards-based content via hands-on, discovery learning. This delivery . . .

- promotes the teaching of Problem Solving in context;
- integrates critical thinking and logical reasoning (Reasoning and Proof);
- provides opportunities for students to explain their thinking (Communication);
- makes Connections across mathematical topics to produce a coherent whole (multiple Content Standards are integrated in each

activity lesson) and across disciplines (such as language arts, science, social studies, music, and art); and

- uses modeling and other forms of Representation (such as graphs, number lines, algebra tiles, and simulations).

The Teacher's Notes for each activity lesson provides a correlation to the Common Core State Standards for Mathematics and to the NCTM *Principles and Standards for School Mathematics*.

Because *Mathematics Explorations* builds on important grade-level mathematical content and connections, the book also aligns with the NCTM *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence* (2006). The integrated curriculum promoted in *Mathematics Explorations* parallels the integrated nature of the curriculum focal points—which draw upon and integrate multiple content topics and process standards in a single focal point.

Motivating Math Learning with Humor

In the May 2006 issue of *Mathematics Teaching in the Middle School* (p. 419), the Editorial Panel promotes "sense of humor" as one of its key suggestions for teaching middle school mathematics. According to the Editorial Panel, "Humor can be used to communicate to students that it is okay for them to look at things in a offbeat way. It is a great way to increase rapport between the teacher and students."
To capture the spirit of motivation, the final chapter in this sourcebook integrates the various Content Standards via mathematics topics that are recreational in nature. To further engage students, humor is used throughout the book in the titles of many activity lessons, in the body of some activity lessons, and as a Mathematical Humor feature with the Teacher's Notes for each lesson. Students will soon see that, in the real world, math can indeed be humorous and fun.

Acknowledgments

I would like to thank my wife, Bonnie, and my children Ben, Jamie, and Joey for all their love, support, and encouragement throughout my career—and in particular, throughout the development of this book. I would also like to thank my sister, Anne Hollenbeck, for creating some of the illustrations used in the book. Finally, I extend a thanks to my editors, Bobbie Dempsey of Good Year Books and my daughter Jamie.

About the Author

David B. Spangler has devoted his entire professional career of more than 35 years to mathematics education. He began as a middle-school mathematics teacher. Later he taught at the community college and university levels, worked as a textbook editor for major publishers, and co-founded a mathematics professional development organization, *Active*Math® Workshops (www.activemath.com). His goal has always been to explore ways to teach mathematics through engaging, real-world applications. He has used many of the activity lessons in this sourcebook with his middle-school students and with teachers in his methods courses and teacher workshops. David's first Good Year Book, *Math for Real Kids,* was recently released in its second edition.

Contents

1 Focus on Number and Operations

In grades 6–8, students should acquire computational fluency —the ability to compute efficiently and accurately— with fractions, decimals, and integers.

—Principles and Standards for School Mathematics (NCTM, 2000)

The Date Detective*

Teacher's Notes

- **NCTM Standards:** Number and Operations
- **CCS Standards:** Operations and Algebraic Thinking; Reason abstractly and quantitatively.
- **Mathematical Topics:** Number Theory (multiples, factors, prime numbers, even and odd numbers, reciprocals, pi, palindromes, perfect numbers)
- **Grouping of Students:** Work independently or in pairs

Background

It's not that I'm so smart, it's just that I stay with problems longer.

—Albert Einstein,
German-born American physicist (1879–1955)

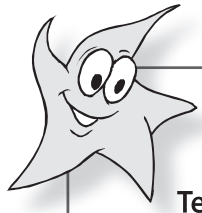
This activity lesson provides an informal way for students to revisit a variety of number theory concepts from the middle-school curriculum. This activity lesson may be used in total, or you may use parts of it as a daily warm-up.

Connecting mathematics concepts to the dates on a calendar reinforces the notion of the "teachable moment." For example, suppose students are learning the concept of "common factors." Suppose also that today's date is, say, 9/30. We could say that today is a *common-factor* date because the month and day numbers have the common factor, 3. You might then ask them to find all common-factor dates during the month of September (or throughout the entire year). Students would thus be reinforcing the day's lesson in a fun, inquisitive way.

Once you permit students to become a "Date Detective," there is no limit to what they are likely to discover! And, they just might enter the classroom each day thinking about mathematics before you even begin your lesson.

For more information on what is special about each of the numbers 1–31, go to http://richardphillips.org.uk/number.

*A source for some of the ideas used in this activity lesson is Larry N. Campbell, "Calendar Search: Every Day Is Special!," edited by David B. Spangler, *Mathematics Teaching in the Middle School* (September 1998), pp. 37–39.



Mathematical Humor

Teacher: There is a difference between having an odd number of students in the room and having a number of odd students in the room. Of course, it's also possible to have an even number of odd students in the room.

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

Johnny: Grandma, can you help me find the least common multiple?

Grandmother: Are they still looking for that? They were looking for that when I was your age!

Solutions

1. 1/1; 2/1, 2/2; 3/1, 3/3; 4/1, 4/2, 4/4; 5/1, 5/5; 6/1, 6/2, 6/3, 6/6; 7/1, 7/7; 8/1, 8/2, 8/4, 8/8; 9/1, 9/3, 9/9; 10/1, 10/2, 10/5, 10/10; 11/1, 11/11; 12/1, 12/2, 12/3, 12/4, 12/6, 12/12

2. May: 5/5, 5/10, 5/15, 5/20, 5/25, 5/30; June: 6/6, 6/12, 6/18, 6/24, 6/30; July: 7/7, 7/14, 7/21, 7/28; August: 8/8, 8/16, 8/24

3. 2/2, 2/3, 2/5, 2/7, 2/11, 2/13, 2/17, 2/19, 2/23, 2/29; 3/2. 3/3, 3/5, 3/7, 3/11, 3/13, 3/17, 3/19, 3/23, 3/29, 3/31; 5/2, 5/3, 5/5, 5/7, 5/11, 5/13, 5/17, 5/19, 5/23, 5/29, 5/31; 7/2, 7/3, 7/5, 7/7, 7/11, 7/13, 7/17, 7/19, 7/23, 7/29, 7/31; 11/2, 11/3, 11/5, 11/7, 11/11, 11/13, 11/17, 11/19, 11/23, 11/29

4. The remaining *square-me* dates are 1/1, 2/4, and 3/9, 4/16.

5a. 3/14 (As an interesting sidelight, March 14 is the birthday of Albert Einstein, born in 1879.)

5b. 3/14/15 at 9:26 A.M. or P.M. (since $\pi = 3.1415926\ldots$)
Note: If a rounded value of π is used, the time would be 9:27 (because $\pi = 3.14159265\ldots$, or 3.1415927, to the nearest hundred-millionth).

5c. March 14, 1592 (or perhaps March 14, 1593, if the date is not truncated).

6. 9/9/99 (The next one will be on 1/1/11.)

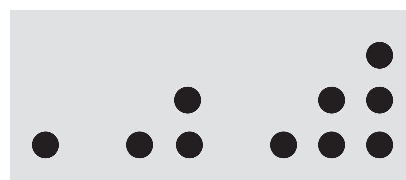
7. Possible answers include any ten of these pairs: 3/1 & 1/3, 2/6, 3/9, 4/12, 5/15, 6/18, 7/21, 8/24, 9/27, 10/30; 3/2 & 2/3, 4/6, 6/9, 8/12, 10/15, 12/18; 3/3 & 1/1, 2/2, 4/4, 5/5, 6/6, 7/7, 8/8, 9/9, 10/10, 11/11, 12/12; 3/4 & 4/3, 8/6, 12/9; 3/5 & 5/3, 10/6; 3/6 & 2/1, 4/2, 6/3, 8/4, 10/5, 12/6; 3/7 & 7/3; 3/8 & 8/3; 3/9 & 3/1, 6/2, 9/3, 12/4; 3/10 & 10/3; 3/11 & 11/3; 3/12 & 4/1, 8/2, 12/3; 3/15 & 5/1, 10/2; 3/18 & 6/1, 12/2; 3/21 & 7/1; 3/24 & 8/1; 3/27 & 9/1; 3/30 & 1/10

8a. 10/02/2001

8b. Previous palindromic date: 08/31/1380
If leading zeros are NOT used in months/days, but 2-digit years ARE used, there are at least six more palindromic dates: 10/11/01, 10/22/01, 11/11/11, 11/22/11, 12/11/21, 12/22/21.

9. Sample answer: An *odd-number* date can only occur during an odd-numbered month (Jan., March, May, July, Sept., and Nov.). The four 31-day months in that group each have 16 *odd-number* dates. The other two odd-numbered months each have 15 *odd-number* dates. So each year there are 64 + 30, or 94, *odd-number* dates.

10. 6/6 and 6/28
11. Answers will vary. Possible special dates might include the following:
    - **composite** dates: Both the month and day numbers are composite numbers. The composite dates are selected dates in April (4/4, 4/6, 4/8, 4/9, 4/10, etc.), June (6/4, 6/6, 6/8, 6/9, etc.), August (8/4, 8/6, etc.), September (9/4, 9/6, etc.), October (10/ 6, 1/8, etc.), and December (12/4, 12/6, etc.).
    - **prime/composite** dates: The month is a prime number; the day is a composite number. Examples include 2/4, 2/6, 2/9, and so on.
    - **composite/prime** dates: The month is a composite number; the day is a prime number. Examples include 4/2, 4/3, 4/5, and so on.
    - **neither prime nor composite** date: 1/1 (Because 0 and 1 are the only numbers that are neither prime nor composite, 1/1 is the only such date.)
    - **relatively prime** dates: Two numbers whose only common factor is 1 are *relatively prime* because their only common factor is 1. So, 6/11 is a *relatively prime* date. Note that a *relatively prime* date can be formed by two prime numbers (as in 11/2), two composite numbers (as in 9/10), or by one of each (as in 12/7). There is a string of 33 consecutive *relatively prime* dates from 12/31 through 2/1.
    - **double-pi** day: 6/28 (2 × 3.14 = 6.28)
    - **half-day** dates: The month number is half the day number: 1/2, 2/4, 3/6, 4/8, 5/10, 6/12, 7/14, 8/16, 9/18, 10/20, 11/22, and 12/24.
    - **odd-digit** dates: If all four digits are used for the year, the next "all odd digit date" will take place in more than 1,000 years on 1/1/3111. The most recent one was on 11/19/1999.
    - **even-digit** dates: If all 4 digits are used for the year, the last "all even digit date" prior to the year 2000 occurred back on 8/28/888. The year 2008 has many of them, such as 2/2/08, 2/4/08, and so on.
    - **square number** dates (month/day): The square numbers are 1, 4, 9, 25, and so on. So the square-number dates are 1/1, 1/4 1/9, 1/16, 1/25; 4/1, 4/4, 4/9, 4/16, 4/25; 9/1, 9/9, 9/4, 9/16, and 9/25.
- **triangular number** dates (month day): The triangular numbers are 1, 3, 6, 10, 15, and so on, because dots representing those numbers can be put in the shape of a triangle as shown.



The triangular number dates are:

1/1, 1/3, 1/6, 1/10, 1/15, 1/21, 1/28;
3/1, 3/3, 3/6, 3/10, 3/15, 3/21, 3/28;
6/1, 6/3, 6/6, 6/10, 6/15, 6/21, 6/28;
10/1, 10/3, 10/6, 10/10, 10/15, 10/21, 10/28

- **cube-me** dates: The day number is the cube of the month number. The cube-me dates are 1/1, 2/8, and 3/27.
- **square root** dates: The day number is the square root of the month number. The square root dates are 1/1, 4/2, and 9/3.
- **cube root** dates: The day number is the cube root of the month number. The cube root dates are 1/1 and 8/2.
- **year-product** dates: The year number is the product of the month number and day number. The *year-product* dates from 1996 through 1999 were: 8/12/96, 12/8/96, 7/14/98, 9/11/99, and 11/9/99.

In 2006, *year-product* dates occurred on 1/6/06, 2/3/06, 3/2/06, and 6/1/06. **Question:** Which year (during any century) has the most *year-product* dates? (The year ending in '24 has seven: 1/24/24, 2/12/24, 3/8/24, 4/6/24, 6/4/24, 8/3/24, and 12/2/24.)

- **consecutive digits** dates: In a given century, the following dates (month/day/year) consist of consecutive digits: 8/9/01 (the last one to take place, if you accept that 0 comes after 9), 1/2/34, 2/3/45, 3/4/56, 5/6/78, and 6/7/89.
- **consecutive odd-number** dates: In any given century, the following dates (month/day/year) consist of consecutive odd numbers: 1/3/05, 3/5/07, 5/7/09, and 7/9/11.
- **consecutive even-number** dates: In any given century, the following dates (month/day/year) consist of consecutive even numbers: 2/4/06, 4/6/8, 6/8/10, and 8/10/12.
- **consecutive numbers** dates: 1/2/03, 2/3/04, 3/4/05, 4/5/06, 5/6/07, 6/7/08, 7/8/09, 8/9/10, 9/10/11, 10/11/12, 11/12/13, and 12/13/14.
- **power-year** dates: The next date where the month raised to the power of the day yields the year (all four digits) is 2/11/2048.
- **consecutive Fibonacci numbers** dates: The numbers 1, 1, 2, 3, 5, 8, 13, 21, . . . , are called Fibonacci numbers. The Italian mathematician Leonardo Fibonacci (1175?–1240?) wrote about this sequence of numbers in 1202. Each term after the second 1 is the sum of the two previous terms. Dates that consist of consecutive Fibonacci numbers are 1/1, 1/2, 2/3, 3/5, 5/8, and 8/13. Chapter 5 includes an activity lesson devoted to the exploration of Fibonacci numbers.

Extension

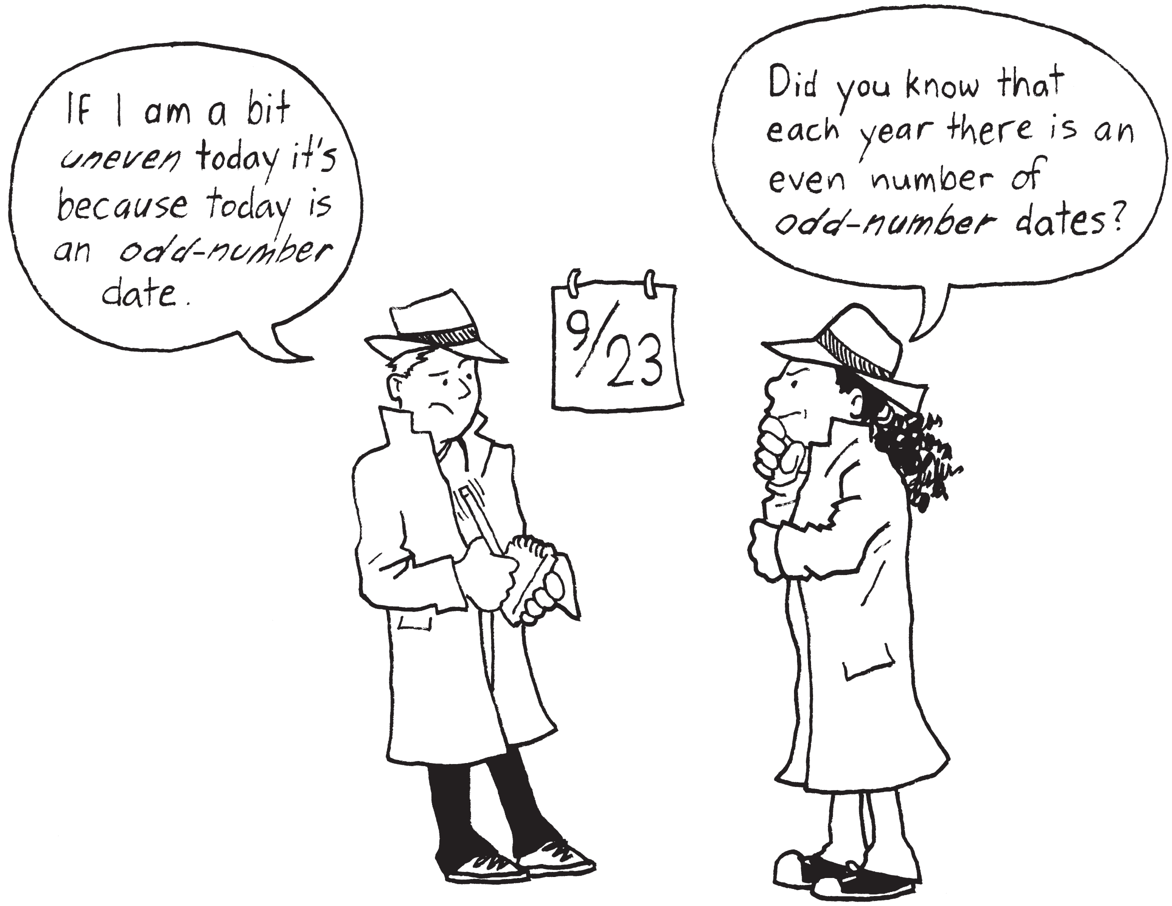
Ask students to discover as many mathematically special things about their birthdays as they can. You might also ask student to try to discover something mathematically special for each day of one month or even for the whole year!

Name ____ Date ___________________

The Date Detective

Take a moment to think about today's date. Is there anything *mathematically special* about it? For example, if today's date were 3/5/08, you could say that this is "sum date"—because the sum of the month and day numbers is equal to the year number. If today's date were 12/4/08, we could say that "today makes a difference"—because the *difference* between the month and the day numbers is equal to the year number.

In this activity lesson, as a "Date Detective," you will apply number theory concepts to explore and uncover mathematical relationships on the calendar. Unless otherwise specified, use only the last two digits for the *years.* Also, unless specified, do not use leading zeros in front of single-digit months/days. OK, let's start. There are dates waiting to be discovered.



Questions

1. Which dates (month/day) are *multiple* dates? Multiple dates are dates where the month number is a multiple of the day number. For example, 9/3 is a multiple date because 9 is a multiple of 3. Another multiple date is 7/1.

____

____

____

2. A *factor* date is one in which the month number is a *factor* of the day number. For example, 9/18 is a *factor* date because 9 is a factor of 18. Another factor date is 1/7. In fact, all dates in January are *factor* dates because 1 is a factor of all numbers. Which dates (month/day) during May, June, July, and August are *factor* dates?

____

____

____

3. A *prime number* is a whole number greater than 1 that has only itself and 1 as factors. The first five prime numbers are 2, 3, 5, 7, and 11.

   Which dates (month/day) are *prime* dates? An example of a prime date is 2/3 because both the month and day numbers are prime numbers.

____

____

____

4. There are five *square-me* dates. One of them is 5/25, because the *square* of the month number is the day number. Name the other four square-me dates.

____

5. *Pi* (π) is the ratio of the circumference to the diameter of any circle. As a decimal, it never ends. $\pi = 3.141592653 \ldots$

   a. Which date each year (month/day) is *pi* day? ____________

   b. Which date and time (month/day/year/time) is *pi* time? ____________

   c. Which date (month/day/4-digit year) was the ultimate *pi* date? ____________

6. When was the last date (month/day/year) *in which all digits were the same?* ____________

7. Two numbers are *reciprocals* if their product is 1. The fractions (dates) 3/7 and 7/3 are reciprocals. Note that pairs of fractions (dates) such as 2/4 and 12/6 are also reciprocals (because $\frac{2}{4} \times \frac{12}{6} = 1$).

   List ten of the pairs of *reciprocal* dates that occur during the month of March.

   ____

   ____

   ____

8. A number that reads the same both forward and backward is a *palindrome.*

   For this problem, use leading zeros in single-digit months/days. Also, use all four digits for the year:

   a. When was the first *palindromic* date (month/day/year) during the twenty-first century? (The twenty-first century began on 01/01/2001.) ___________

   b. When was the last *palindromic* date previous to that one? (Hint: It occurred during the fourteenth century.) ___________

9. Review the cartoon at the beginning of this activity lesson. Then verify that there is indeed an even number of *odd-number* dates each year. (A date such as 9/23 is an *odd-number* date because both 9 and 23 are odd numbers.)

   ____

   ____

   ____

10. A *perfect number* is a whole number *n* for which the sum of all its factors, excluding *n,* is equal to *n.* An example of a perfect number is 6. Its factors, less than 6, are 1, 2, and 3. Since 1 + 2 + 3 = 6, the number 6 is perfect.

    Which two dates (month/day) are the only totally *perfect* dates?______________________

    ____

11. Make up your own types of mathematically special dates.

    ____

    ____

    ____

    ____

    ____

Editor for a Day Error Search*

Teacher's Notes

■	**NCTM Standards:**	Number and Operations
■	**CCS Standards:**	Number and Operations—Fractions; Make sense of problems and persevere in solving them.
■	**Mathematical Topics:**	Multiplication and division of fractions
■	**Grouping of Students:**	Work in pairs or independently. You may consider pairing a student whose strong suit is language arts with one whose strong suit is mathematics.

Background

Ours is not to reason why; just invert and multiply.

—Rhyme often used to help students remember the algorithm for dividing fractions (For an explanation as to why we invert and multiply, see the Background notes below.)

Through the error search, this activity lesson reinforces multiplication and division of fractions—while at the same time integrating language arts skills. Key to the purpose of the activity lesson is the notion that being mathematically literate includes the ability to apply language arts skills in a mathematical setting—and that one should not view disciplines as isolated, unrelated subjects.

Students often enjoy playing the role of a teacher, and in particular, correcting errors made by "someone else." As such, students may become so engrossed in uncovering the errors that they may not be cognizant of the "disguised drill" nature of this activity lesson.

*This activity lesson is based on a "Mathematics Detective" article written by David B. Spangler for the February 2005 issue of *Mathematics Teaching in the Middle School,* copyright 2005, National Council of Teachers of Mathematics. It is reprinted here by permission of the National Council of Teachers of Mathematics. All rights reserved.

This activity lesson is intended to provide the following benefits to students:

- Reinforce editing and proofreading skills. (Some students may discover that editorial work might be a career that would be of interest to them.)
- Encourage students to *think* about what they read.
- Help students become more discriminating readers of written problems.
- Provide *alternative assessment.* Being able to see that something is wrong often only occurs if the basic processes have been understood.

Students often ask, "Why do we invert when we divide, but we don't invert when we multiply?" It is interesting to note that you *can* divide fractions *without* inverting and multiplying, as shown by the examples below. In fact, we can divide fractions by dividing numerators and dividing denominators.

$$\frac{12}{25} \div \frac{3}{5} = \frac{12 \div 3}{25 \div 5} = \frac{4}{5}$$

$$\frac{12}{3} \div \frac{2}{3} = \frac{12 \div 2}{3 \div 3} = \frac{6}{1} = 6$$

Students can check the results with the standard algorithm to see that the quotients are indeed correct. The examples work nicely because the numerators and denominators of the fractions being divided are *divisible* by the respective numerators and denominators of the second fractions. Here is what we can do when that is *not* the case:

Find $\frac{3}{7} \div \frac{2}{3}$. $\frac{3}{7} \div \frac{2}{3} = \frac{3 \div 2}{7 \div 3} = \frac{\frac{3}{2}}{\frac{7}{3}}$

To simplify this *complex fraction,* multiply the numerator and the denominator of the complex fraction by $\frac{3}{7}$. This does not change the complex fraction's value, since multiplying its numerator and denominator by $\frac{3}{7}$ has the effect of multiplying by 1. We choose $\frac{3}{7}$, since $\frac{3}{7} \times \frac{7}{3}$ will give us 1 in the denominator of the complex fraction.

$$\frac{\frac{3}{2}}{\frac{7}{3}} = \frac{\frac{3}{7}}{\frac{3}{7}} \times \frac{\frac{3}{2}}{\frac{7}{3}} = \frac{\frac{3}{7} \times \frac{3}{2}}{\frac{3}{7} \times \frac{7}{3}} = \frac{\frac{3}{7} \times \frac{3}{2}}{1} = \mathbf{\frac{3}{7} \times \frac{3}{2} = \frac{9}{14}}$$

Do you notice something about this result?

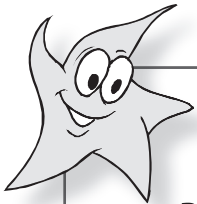
As demonstrated by the above, the "invert and multiply" algorithm is really just a shortcut for finding the quotient of two fractions (and it is helpful in avoiding messy computations as shown above). Hence, "invert and multiply" is convenient—but it is not the only way to divide fractions.

For an error-search lesson similar to this one based on multiplication and division facts, see "Editor for a Day" in *Math for Real Kids,* published by Good Year Books (2005).

Extension

In the first sixteen problems of this activity lesson, there are nine wrong answers. Each wrong answer was obtained due to an incorrect computational procedure. Ask your students to try to discover the computational "error pattern" that was used in each of those problems. Possible error patterns are listed below. Other explanations are possible.

2. The answer was not simplified.
3. Cross-products were computed (4 × 1 and 3 × 5), with one cross-product written as the numerator (4) and the other as the denominator (15).
4. The whole number (4) was multiplied by both the numerator (2) and the denominator (7) of the fraction.
6. Instead of multiplying, a procedure for converting a mixed number to an improper fraction was used. The whole number (5) was multiplied by the denominator (4). This product was added to the numerator (5) and placed over the denominator (4).
7. Diagonal numerators and denominators (2 and 4; 3 and 3) were simplified prior to inverting the divisor.
9. The whole number (5) was divided by the numerator (5) of the divisor prior to inverting the divisor.
11. The dividend was inverted instead of the divisor.
15. The two numerators (3 and 9) and the two denominators (10 and 2) were simplified.
16. $\frac{3}{10}$ was multiplied by 7 instead of being divided by 7. (The divisor, 7, was not inverted.)



Mathematical Humor

Parent: What did you study in math class today?
Student: Fractions.

Parent: What did you learn about fractions?
Student: A fraction of what I was supposed to.

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

Question: Why did the fraction $\frac{1}{5}$ do poorly on a math quiz?
Answer: Because he was *too tense.*

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

A recent newspaper story featured examples of employee complaints in the workforce about customer ignorance. One of the examples came from an employee who works at a hamburger restaurant that displays a sign like this:

1/2 lb. Burger
1/3 lb. Burger
1/4 lb. Burger

The employee has had frequent arguments with customers that are similar to this:

Customer: I ordered the "1 slash 4 Burger." He only ordered the "1 slash 2 Burger." Why is his burger much bigger than mine? Isn't 4 larger than 2?

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

The following quote is probably more thought-provoking than it is humorous:

A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator, the smaller the fraction.

—Leo Tolstoy, Russian novelist (1828–1910)

Solutions

Students should make the corrections directly on the "manuscript" as shown below.

Editor for a Day Error Search

This page of math problems contains many errors. Pretend you are a math editor. ~~You're~~ Your job is to find and correct the errors. In ~~edition~~ addition to math errors, there are errors in spelling, gramm~~e~~ar, punctuation, and style. Have you already found some errors?

The answers to the problems are given in bold type. Good luck in your error search.

Multiply or divide. Simplify your answers.

1. $\frac{1}{2} \times \frac{1}{2} = \mathbf{\frac{1}{4}}$
2. $\frac{1}{3} \times \frac{3}{5} = \mathbf{\frac{3}{15}} = \frac{1}{5}$
3. $\frac{3}{4} \times \frac{1}{5} = \cancel{\mathbf{\frac{4}{15}}}\ \frac{3}{20}$
4. $4 \times \frac{2}{7} = \mathbf{\frac{8}{\cancel{28}}}_{7} = 1\frac{1}{7}$
5. $\frac{2}{3} \div \frac{2}{3} = \mathbf{1}$
6. $5 \times \frac{3}{4} = \mathbf{\frac{\cancel{23}\ 15}{4}} = 3\frac{3}{4}$
7. $\frac{2}{3} \div \frac{3}{4} = \cancel{\mathbf{\frac{1}{2}}}\ \frac{8}{9}$
8. $\frac{1}{8} \div 8 = \mathbf{\frac{1}{64}}$
9. $5 \div \frac{5}{6} = \cancel{\mathbf{\frac{1}{6}}}\ 6$
10. $\frac{9}{10} \times \frac{5}{6} = \mathbf{\frac{3}{4}}$
11. $\frac{3}{4} \div \frac{1}{8} = \cancel{\mathbf{\frac{1}{6}}}\ 6$
12. $0 \times \frac{3}{5} = \mathbf{0}$

~~14.~~ 13. $\frac{4}{3} \times 4 = \mathbf{5\frac{1}{3}}$

~~15.~~ 14. $\frac{3}{10} \times \frac{9}{2} = \cancel{\mathbf{\frac{3}{5}}}\ 1\frac{7}{20}$

~~16.~~ 15. $\frac{3}{10} \div 7 = \cancel{\mathbf{2\frac{1}{10}}}\ \frac{3}{70}$

~~17.~~ 16. $9 \div 18 = \mathbf{\frac{1}{2}}$

Solve each problem.

17. The lengt h of a track around a football field is $\frac{1}{4}$ mile~~s~~. You jog 6 times around the track. How far do you jog? — $\mathbf{1\frac{1}{2}}$ miles

18. Arbor School invited boys to try out for it~~'~~s basketball team. Of the 36 boys who ~~tryed~~ tried out, $\frac{1}{3}$ made the team. How many boys did not make ~~there~~ their team? — ~~**12**~~ 24 **boys**

19. Doug ate $\frac{3}{4}$ of a pie. Then he ate $\frac{1}{4}$ of what was left. ~~left~~. How much of the pie did ~~s~~he eat? — $\mathbf{\frac{\cancel{15}\ 13}{16}}$ **of the pie**

20. Miss Smith's class is cutting ribbon. How many $\frac{1}{2}$-inch strips can be cut from 20 inches of ribbon? — ~~**10**~~ 40 **strips**

Name ____ Date ___________________

Editor for a Day Error Search

This page of math problems contains many errors. Pretend you are a math editor. You're job is to find and correct the errors. In edition to math errors, there are errors in spelling, grammer punctuation and style. Have you already found some errors?

The answers to the problems are given in bold type. Good luck in your error search.

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5. $\frac{2}{3} \div \frac{2}{3} = \mathbf{1}$
6. $5 \times \frac{3}{4} = \mathbf{\frac{23}{4}}$
7. $\frac{2}{3} \div \frac{3}{4} = \mathbf{\frac{1}{2}}$
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12. $0 \times \frac{3}{5} = \mathbf{0}$
14. $\frac{4}{3} \times 4 = \mathbf{5\frac{1}{3}}$
15. $\frac{3}{10} \times \frac{9}{2} = \mathbf{\frac{3}{5}}$
16. $\frac{3}{10} \div 7 = \mathbf{2\frac{1}{10}}$
17. $9 \div 18 = \frac{1}{2}$

Solve each problem

17. The lengh of a track around a football field is $\frac{1}{4}$ miles. You jog 6 times around the track. How far do you jog? — **$1\frac{1}{2}$**

18. Arbor School invited boys to try out for it's basketball team. Of the 36 boys who tryed out, $\frac{1}{3}$ made the team. How many boys did not make there team? — **12 boys**

19. Doug ate $\frac{3}{4}$ of a pie. Then he ate $\frac{1}{4}$ of what was left left. How much of the pie did she eat? — **$\frac{15}{16}$ of the pie**

20. Miss Smiths class is cutting ribbon How many $\frac{1}{2}$-inch strips can be cut from 20inches of ribbon. — 10 strips

THE PRICE *ISN'T* RIGHT*

TEACHER'S NOTES

- **NCTM Standards:** Number and Operations
- **CCS Standards:** Ratios and Proportional Relationships; Reason abstractly and quantitatively. Attend to precision.
- **Mathematical Topics:** Find percent discount; solve problems involving decimal applications
- **Grouping of Students:** Work in pairs or individually

BACKGROUND

This activity lesson provides students with opportunities to uncover mathematical errors embedded in real-world situations. In fact, each error was experienced by the author.

The first situation presents a common misconception related to percents. In this situation, the percent discount (20%) for each meal is erroneously *added* for each of two people so that a 40% discount is applied to the total bill—instead of the 20% being applied to the total bill. The second situation involves the difficulty some people have in writing a decimal for amounts of money that involve a fraction of a cent (in this case, 4.6¢). This problem also highlights the use of a pricing structure that is probably more advantageous to the customer than it should be.

*This activity lesson is based on a "Mathematics Detective" article written by David B. Spangler for the November 1999 issue of *Mathematics Teaching in the Middle School,* copyright 1999, National Council of Teachers of Mathematics. It is reprinted here by permission of the National Council of Teachers of Mathematics. All rights reserved.

MATHEMATICAL HUMOR

Both examples in this activity lesson highlight a situation in which a business establishment was being "too generous" with its customers. These examples bring to mind a story about a sales meeting that was being held by a particular company. During the meeting, the president announced to his sales force, "I have bad news and good news to report. The bad news is that we're losing money on every item we sell. But the good news is that business has been lousy."

SOLUTIONS

1. a. \$10.36 b. \$11.60
   c. \$21.96 d. \$16.47
2. Mr. And Mrs. Lucky were indeed *lucky.* They should have received a 20% discount for each of two dinners, or 20% off the sum of the two regular prices. So, as demonstrated by the distributive property below, this is not equal to 40% off the total of the two regular prices.

Amount of discount on each of two dinners:		Amount of discount on the total of the two dinners:
0.20(\$12.95) + 0.20(\$14.50)	=	0.20(\$12.95 +\$14.50)
\$2.59 + \$2.90	=	0.20(\$27.45)
\$5.49	=	\$5.49

So, the discounted price for the two dinners should have been \$27.45 – \$5.49, or \$21.96. (The waiter and manager wanted them to pay \$27.45 – 0.40(\$27.45) = \$27.45 – \$10.98, or \$16.47.) In general, if x and y are the regular prices of any two dinners, respectively, then $0.20x + 0.20y = 0.20(x + y)$.

3. The waiter and manager evidently figured that the percent discount that is applied to the total bill increases by 20% with each customer. Mrs. Lucky probably figured that by applying this incorrect reasoning to five dinners, the waiter and manager would realize their error—since the total bill for five dinners (based on this incorrect logic) would be reduced by 100%.

4a. \$230.05

b. The customer was charged \$0.46 per copy instead of \$0.046.

5a. He probably figured that he could help the clerk to mentally compute the cost of 5,000 copies at 5¢ per page to be \$250.

b. Answers will vary. Mr. Inkblott probably attempted to say the following: "If 1 copy costs 5¢, then you would get 20 copies for \$1. This means that you would get 100 copies for \$5, and you would get 1,000 copies for \$50. Thus, 5,000 copies would cost \$250. Since the actual price is only 4.6¢ per copy, I should pay less than \$250. I certainly should not pay more than \$2,000!"

6a. The total cost for 5,001 copies is actually less than the total cost for 4,800 copies. The cost for 4,800 copies (at 7¢ per page) is \$336; the cost for 5,001 copies (at 4.6¢ per page) is \$230.05. So Mr. Inkblott will save \$105.95 by having an extra 201 copies made.

b. No. The cost for 7,501 copies (at 3.5¢ per page) is \$262.54. Since Mr. Inkblott doesn't need so many copies, he is better off going with the cheapest price (5,001 copies for \$230.05).

7. Answers will vary. From the store's point of view, it really does not make sense to print more total copies for less total money. A pricing structure such as the one shown below might make more sense for volume discounts. With this structure, the discounts take effect incrementally—and are thus not applied to all of the copies.

Number of Copies	Price per Copy
First 5,000 copies	7¢ each
Next 2,500 copies	4.6¢ each
All copies above 7,500	3.5¢ each

EXTENSION

Ask students to become "mathematics detectives" in their neighborhood by going on a pursuit for misleading or mathematically incorrect ads. Such ads may be found in newspapers, direct mail promotions, on television, in stores, and so on. Students should also be on the alert for situations in which employees at a place of business incorrectly use information that appears in store advertisements. Encourage students to bring the ads and stories to class for discussion.

Name____Date ___________________

The Price *Isn't* Right

The stories you are about to read are true; only the names have been changed. In each case, employees charged customers incorrect amounts due to mathematical errors. So, "come on down" and let's investigate what happened. And let's also do some mathematics to make sure that the price *is* right.

The Senior Citizen Special

Mr. and Mrs. Lucky, who are senior citizens, ate dinner at the restaurant that displayed the sign shown below. Let's pick up the conversation at the end of their dinner.

Senior Citizen Special
Dinners
20% OFF

Waiter: Is there anything else I can get you?

Mr. Lucky: No thank you; I think everything is fine. All we need is our check.

Waiter: Well, let's see . . . there's two of you—so that means you get 40% off.

Mrs. Lucky: Excuse me sir, I think we should only get 20% off the total bill.

Waiter: I think you get 40% off, but let me check with the manager.

The waiter goes out and returns with the manager.

Manager: Your waiter is correct. You're entitled to 40% off the total check.

Mrs. Lucky: Well, if that's the case, the next time my husband and I come here we'll bring three of our friends so that all of us will eat for free!

Questions

1. Suppose the regular price for Mrs. Lucky's dinner was $12.95, and the regular price for Mr. Lucky's dinner was $14.50.
   a. How much would Mrs. Lucky's dinner cost at 20% off? ___________
   b. How much would Mr. Lucky's dinner cost at 20% off? ___________
   c. What would be the total cost for the two dinners at 20% off each dinner? ___________
   d. What would be the total cost for the two dinners at 40% off the total of the regular prices? ___________
2. Were Mr. and Mrs. Lucky *lucky,* or were the waiter and manager correct? Explain your reasoning.

   ____

   ____

3. Why do you think Mrs. Lucky suggested to the waiter and manager that the next time they come to the restaurant they might bring three of their friends with them?

   ____

   ____

Purchasing Photocopies at Copies, Ink

Mr. Inkblott goes to Copies, Ink, to have some photocopies made. He needs 4,800. After examining the sign, he decides to have 5,001 copies made (even though he doesn't need that many). Later that day he returns to Copies, Ink, to pick up his order. Let's pick up the conversation as Mr. Inkblott hands the clerk his credit card.

Copies, Ink

If you order . . .	You pay . . .
up to 5,000	7¢ each
5,001 to 7,500	4.6¢ each
more than 7,500	3.5¢ each

Clerk: This is a good-sized order.

Mr. Inkblott: Yeah, and I made it even bigger because of your great prices here.

Clerk: The total comes to $2,300.46.

Mr. Inkblott: What? That's impossible! There has to be an error here!

Clerk: I'm sorry, but the cash register shows $2,300.46.

Mr. Inkblott: Wait, let me try to explain your error. Let's suppose there were 5,000 copies at 5¢ per page.

Clerk: But the price is 4.6¢ per page. Don't try to confuse me.

Questions

4a. Based on the sign, how much should 5,001 copies cost? ___________

b. What error was made in determining the total cost?

____

____

5a. Why did Mr. Inkblott attempt to explain the cost for 5,000 copies at 5¢ per page?

____

____

b. How would you have tried to explain the error?

____

____

6a. Why did Mr. Inkblott order 5,001 copies instead of 4,800 copies?

____

____

b. Would Mr. Inkblott have been even better off having 7,501 copies made? Explain.

____

____

7. Although many types of stores offer discounts for large quantities, does the pricing structure at Copies, Ink, really make sense? From the point of view of Copies, Ink, how might the pricing structure for volume discounts be improved?

____

____

____

____

____

Calculations That Are NOT *Par for the Course**

Teacher's Notes

- **NCTM Standards:** Number and Operations
- **CCS Standards:** Ratios and Proportional Relationships; Expressions and Equations; Look for and make use of structure.
- **Mathematical Topics:** Percent change; mental math; use trial and error; look for a pattern
- **Grouping of Students:** Work in pairs or individually

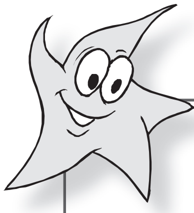
Background

The activity lesson assumes prior student knowledge in the area of computing percent change and rounding percents. The activity lesson is based on an actual news article that appeared in a major city newspaper in the United States. It illustrates the importance of being able to accurately communicate quantitative information. In particular, it shows that journalists need to have good analytical skills (as well as good verbal skills)—because data are frequently used to quantify a situation.

Some students may have difficulty correcting the percent increases for George Dunne (South) because those increases involve percents greater than 100%. Advise students that an increase from \$6 to \$12 represents a 100% increase because the amount of increase (\$6) is equal to the original amount. So an increase from \$6 to \$18 represents a 100 + 100, or 200%, increase. Thus, an increase from \$6 to \$20 represents an additional $\frac{2}{6}$ increase (33%) above that, or 233% in all.

Encourage students to use trial-and-error when they try to discover the error pattern asked for in question 4a.

*This activity lesson is based on a "Mathematics Detective" article written by David B. Spangler for the January 1998 issue of *Mathematics Teaching in the Middle School,* copyright 1998, National Council of Teachers of Mathematics. It is reprinted here by permission of the National Council of Teachers of Mathematics. All rights reserved.



Mathematical Humor

The difficulty in calculating percent increase brings to mind a story about a seventh-grade student who was not very good in math. Eventually, he graduated from high school and went into business—where he made millions and millions of dollars. One day, he went back to visit his former seventh-grade teacher. He pulled up to the school in a fancy car and was wearing expensive clothing. "Young man, you appear to be a real success," said the startled teacher. "When you were in my class, you couldn't add 2 and 2. What is your secret?" asked the teacher. The young man responded: "Everything changed after high school. I learned that you can go into business and buy something for $1, and then sell it for $4. And you know something, it's that extra *3 percent* that makes all the difference!"

Solutions

1a. "CHANGE" refers to the amount of increase, in dollars, from 1988 to 1992 at each course.

b. Yes. The change in weekday and weekend fees at Kemper Lakes should be $25, not $35.

2a. "(PCT.)" refers to the percent change from 1988 to 1992 at each course.

b. Parentheses usually refer to decreases, or negative values. They are often used in financial statements instead of negative signs. Parentheses should not be used to show percent increases.

3a. Most students should be able to use mental math or estimation to determine that the percent increases given in the table for Village Links (18%), Dubsdread (50%), and George Dunne (33% and 35%) are far too low. Those percents are not even in the "driving range" of the correct percents.

b. A corrected table is given at right. Note that every percent increase in the original table was computed incorrectly.

Course	Percent Increase
Jackson Park (City)	
Weekday fee	17%
Weekend fee	23%
Glencoe (North)	
Weekday fee	47%
Weekend fee	41%
Kemper Lakes (Northwest)	
Weekday fee	38%
Weekend fee	38%
Village Links (West)	
Weekday fee	83%
Weekend fee	84%
Dubsdread No. 4 at Cog Hill (SW)	
Weekday fee	100%
Weekend fee	100%
George Dunne (South)	
Weekday fee	233%
Weekend fee	257%

c. Methods may vary. Many students will likely first subtract the 1988 cost from the 1992 cost to determine the amount of change. (These values are given in the "CHANGE" column.) Then they will divide the amount of change by the 1988 cost to determine the percent increase.

4a. It appears that the reporter divided the 1992 cost by the 1988 cost but then misinterpreted the result. For example, for Jackson Park weekday, $7 \div 6$ yields $1.1\overline{6}$. It appears that the reporter "converted" this result to 11%. For Jackson Park weekend, $8 \div 6.50$ yields about 1.23—which evidently was "converted" to 12%. As indicated in question 4c, in some cases the reporter apparently rounded up the results (as in Glencoe's weekday: $22 \div 15 = 1.4\overline{6}$, yielding 15%).

It is interesting to note that dividing the 1992 cost by the 1988 cost could provide a "fairway" to the correct results. To find the percent increase for Jackson Park weekday, you can divide 7 by 6. The result, $1.1\overline{6}$, however, should be interpreted as meaning the 1992 cost is about 117% of the 1988 cost. Thus, the percent increase is 17%. This method, however, is difficult for some students to conceptualize—especially when they work with percent increases over 100%. For example, using this method for George Dunne weekday yields: $20 \div 6 = 3.3\overline{3}$. The 1992 cost is thus about 333% of the 1988 cost—representing a 233% increase over the 1988 cost.

b. After performing the calculation $70 \div 35 = 2$, the reporter might have wondered what to do with the 2. Perhaps the reporter confused the 2 with $\frac{1}{2}$, thus obtaining a percent increase of 50%. The correct interpretation of the 2 is that the 1992 cost is 200% of the 1988 cost, yielding a percent increase of 100%. Another possibility is that the reporter, not knowing what to do with the 2, switched gears and divided 35 by 70—yielding 0.50—thus obtaining the 50%.

c. Probably not. The errors do not all fit the same pattern. For example, the computation for Jackson Park's weekday fee is $7 \div 6 = 1.1\overline{6}$, but the reporter gave a percent increase of 11% (rather than a "rounded up" value of 12%). Yet, the quotient for Glencoe's weekday fee, $22 \div 15$, or $1.4\overline{6}$, *was* "rounded up" to 15%.

5. No. The increase from \$18 to \$33 is about 45%. Even the incorrect percent given in the table (18%, not the 10% mentioned in the article) is probably not a "moderate" increase.

6. Yes. Charts, tables, graphs, and other numerical data are an important part of most news articles. Journalists need to have good analytical skills—as well as good verbal skills—because graphical displays and numerical data can paint a picture that words alone cannot describe. Thus, mathematics clearly can be a *driving force* to help one get to the "green."

Extension

Have students find out what it costs to play golf in the area where they live. Have them compare the costs among several courses. These may include the costs of greens fees, equipment rental, lessons, and so on. Have them report which course is the least expensive, which is the most expensive, and how much more one costs than the other on a percent basis.

Name____ Date ____________________

Calculations That Are NOT *Par for the Course*

A number of years ago, a major city newspaper in the United States reported increases in golf fees in its local area. The table below accompanied the article.

Glance over the data in the table. Can you tell that the table contains incorrect calculations? To use golf lingo, these calculations have been "sliced out of bounds." So, put on your detective hat, and let's search "fore" and chip away at the errors in the table. Then "tee off" on the questions that follow. Our goal is to help the newspaper reporter get out of the "sand traps"—and "iron" out those errors!

IT COSTS MORE, OF COURSE

COURSE	1988	1992	CHANGE	(PCT.)
Jackson Park (City)				
Weekday fee	$6	$7	+$1	(11)
Weekend fee	$6.50	$8	+$1.50	(12)
Glencoe (North)				
Weekday fee	$15	$22	+$7	(15)
Weekend fee	$17	$24	+$7	(14)
Kemper Lakes (Northwest)				
Weekday fee	$65	$90	+$35	(14)
Weekend fee	$65	$90	+$35	(14)
Village Links (West)				
Weekday fee	$18	$33	+$15	(18)
Weekend fee	$19	$35	+$16	(18)
Dubsdread No. 4 at Cog Hill (Southwest)				
Weekday fee	$35	$70	+$35	(50)
Weekend fee	$35	$70	+$35	(50)
George Dunne (South)				
Weekday fee	$6	$20	+$14	(33)
Weekend fee	$7	$25	+$18	(35)

Questions

1a. What does "CHANGE" stand for in this table?

____

b. Are there any errors in the "CHANGE" column?
Hint: There is a "double bogey" at one of the courses.

____

2a. What does "(PCT.)" stand for in this table?

____

b. Do you think parentheses should be used in the "(PCT.)" column of this table? (You might want to check with someone who is familiar with financial statements to find out what parentheses usually mean when they appear in such documents.)

____

3a. Use mental math or estimation to answer this question:
For which courses did the reporter make an error in the "(PCT.)" column?

____

b. Use a calculator to correct all of the errors in the "(PCT.)" column. Write the correct values to the right of the incorrect ones. Note: Be sure to use the correct values you found in question 1b.

c. Explain the method you used to find the correct results.

____

____

____

4a. You should now be in the "swing" to use your clues to do some serious detective work. What do you suppose the newspaper reporter did to arrive at most of the "(PCT.)" results in the table? That is, try to find a pattern that describes most of his or her calculations.

____

____

____

b. The calculations for Dubsdread probably don't fit your pattern. What do you suppose the reporter did to arrive at those results?

____

____

____

c. Do you think the errors in the "(PCT.)" column are the result of a "computer error"? Explain.

____

____

____

5. According to the newspaper article that accompanied the table, "The price of a round of golf at Village Links has gone up a moderate 10 percent to $33 on weekdays." Did the fees increase by a "moderate 10 percent"? Explain.

____

____

____

6. Do journalists need to have basic mathematics skills? Explain.

____

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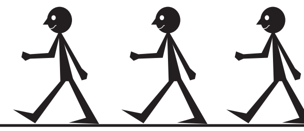
Discovering Integer Rules with Integer Man

Teacher's Notes



- **NCTM Standards:** Number and Operations
- **CCS Standards:** The Number System; Look for and express regularity in repeated reasoning.
- **Mathematical Topics:** Addition and subtraction of integers; look for a pattern
- **Grouping of Students:** Work in pairs or individually

Background

Studying math accomplishes two goals: It prepares some kids to think like scientists, and it prepares all the rest of them to think, period.

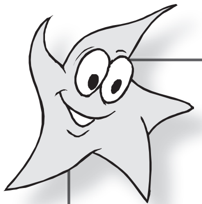
—Marilyn vos Savant, American magazine columnist (1946–), in response to a reader's question about when one would ever compute with integers in real life.

This activity lesson provides guided discovery for students to develop their own rules for adding and subtracting integers. The sets of exercises are set up so that the answers to the subtraction problems are, in order, the same as the answers to corresponding addition problems. The goal is that students will discover and conclude that all subtraction problems can be converted to corresponding addition problems. Hence, students need to learn only one set of rules for the two operations.

The magnitude of the numbers used in this activity lesson is small to encourage students to focus on using a number line to discover a set of rules (rather than getting bogged down in rigorous computations).

When students develop a set of rules for adding integers in question 13, encourage them to consider two cases: (1) when the signs of the two addends are the same, and (2) when the signs are different.

You may want to use a large number line on the floor and have students play the part of Integer Man to act out the problems (going back and forth on the number line).



Mathematical Humor

The following problem appeared in a mathematics book:

Q. Sally began at point 0 on a number line. She walked 5 miles due east. Then she walked 5 miles due west. How many miles did she walk in all?

A. *Answer given in textbook: 0 miles*

Obviously the author of the textbook confused Sally's end point with the total miles Sally traveled. The textbook author obviously would be quite frustrated if he or she ever exercised on a treadmill!

Solutions

1. 2	2. (-3)	3. 0
4. (-5)	5. 4	6. 0
7. (-4)	8. (-7)	9. (-9)
10. (-6)	11. 7	12. (-7)

13. Answers will vary. When the signs are the same: Add the numbers without regard to sign and then use the sign of the numbers for the sum. When the signs are different: Subtract the smaller number from the larger number without regard to signs and then use the sign of the larger number (again ignoring signs) for the sum.

14. 2	15. (-3)	16. 0
17. (-5)	18. 4	19. 0
20. (-4)	21. (-7)	22. (-9)
23. (-6)	24. 7	25. (-7)

26. The answers to questions 1–12 are the same as the answers to questions 14–25, respectively.

27. Answers may vary. Change each subtraction problem to an addition problem by adding the opposite of the number that was to be subtracted.

Extension

Have students apply their rule for adding and subtracting integers where the magnitude of the integers is greater than that used in this activity lesson.

Name______________________________ Date ______________

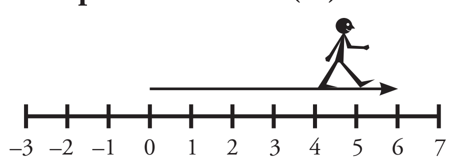
Discovering Integer Rules with Integer Man

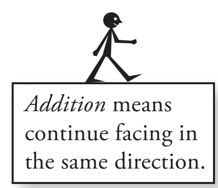
In this activity lesson, you will observe the actions of Integer Man (a little stick figure) walking on a number line. This should help you discover a rule for adding and subtracting integers.

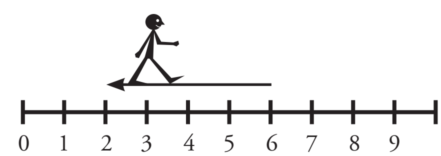
Addition of Integers

▶ For positive numbers, move forward. For negative numbers, move backward.

Example 1: Find 6 + (-4)





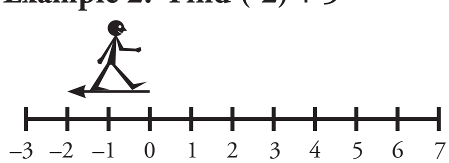


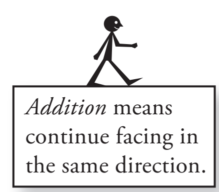
Start at 0, facing the positive numbers.
Walk *forward 6 steps for 6.*

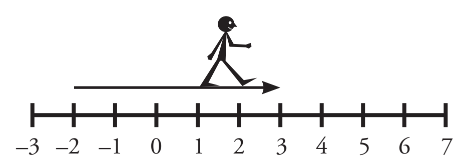
Then walk *backward 4 steps for –4.*
End at 2.

So, 6 + (–4) = 2.

Example 2: Find (-2) + 5







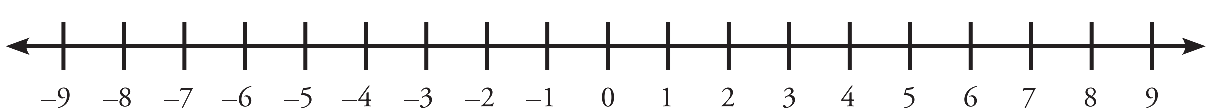
Start at 0, facing the positive numbers.
Walk *backward 2 steps for –2.*

Then walk *forward 5 steps for 5.*
End at 3.

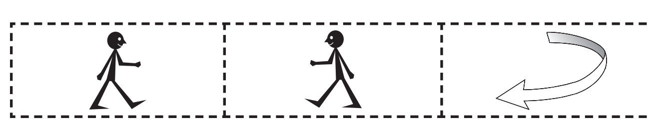
So, (–2) + 5 = 3.

Use the number line and Integer Man below to help you find each sum.

1. 5 + (-3) = ________	2. 4 + (-7) = ________	3. 6 + (-6) = ________
4. (-7) + 2 = ________	5. (-1) + 5 = ________	6. (-8) + 8 = ________
7. (-2) + (-2) = ________	8. (-3) + (-4) = ________	9. (-1) + (-8) = ________
10. 0 + (-6) = ________	11. (-2) + 9 = ________	12. (-5) + (-2) = ________



Cut out these figures for use on the number line.



13. Put on your detective hat. Study the results in questions 1–12. Write a set of rules for finding the sum of two integers.

____

____

____

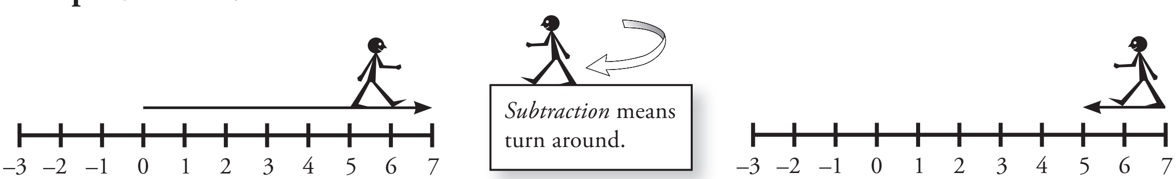
____

____

Subtraction of Integers

▸ For positive numbers, move forward. For negative numbers, move backward.

Example 3: Find 7 – 2.

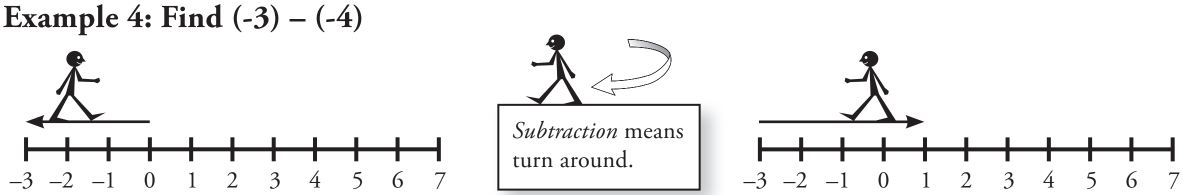


Start at 0, facing the positive numbers.
Walk *forward 7 steps for 7.*

Then walk *forward 2 steps for 2.*
End at 5.

So, 7 – 2 = 5.

Example 4: Find (-3) – (-4)



Start at 0, facing the positive numbers.
Walk *backward 3 steps for –3.*

Then walk *backward 4 steps for –4.*
End at 1.

So, (–3) – (–4) = 1.

Use a number line and Integer Man to help you find each difference.

14. 5 – 3 = ___________	15. 4 – 7 = ____________	16. 6 – 6 = ____________
17. (-7) – (-2) = ________	18. (-1) – (-5) = ________	19. (-8) – (-8) = ________
20. (-2) – 2 = _________	21. (-3) – 4 = __________	22. (-1) – 8 = __________
23. 0 – 6 = ___________	24. (-2) – (-9) = _______	25. (-5) – 2 = ___________

____

26. Compare the answers to questions 1–12 with those in questions 14–25.
What do you notice?

____

27. Write a rule for subtracting two integers.

____

____

Who Is the *Prime* Suspect?
Using Prime Numbers and a Spreadsheet to Decipher a Secret Code

Teacher's Notes

- **NCTM Standards:** Number and Operations
- **CCS Standards:** Operations and Algebraic Thinking; Use appropriate tools strategically.
- **Mathematical Topics:** Prime numbers; divisibility of numbers; division by 0; create a computer spreadsheet to perform simultaneous calculations on a set of numbers
- **Grouping of Students:** Work in pairs or individually

Background

Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the human mind will never penetrate.

—Leonhard Euler,
Swiss mathematician (1707–1783)

Cryptology is the branch of knowledge that deals with secret writing or communications in code. It originated with the human desire to communicate *secretly*—and is as old as writing itself. Very large prime numbers are often used in codes in computer technology to improve the security of computer installations because such prime numbers are very difficult to discover. This is because prime numbers do not occur in a regular sequence.

Since there is no formula to find prime numbers, mathematicians must *search* for them. In 1772, Leonhard Euler showed that the expression $x^2 + x + 41$ produces a prime number for each value of x from 0–39 (41 for $x = 0$; 43 for $x = 1$; 47 for $x = 2$, and so on). Another expression, $x^2 + x + 17$, produces a prime for each value of x from 0–15 (17 for $x = 0$; 19 for $x = 1$; 23 for $x = 2$; and so on). But these formulas are obviously quite limiting.

Governments, banks, and manufacturers primarily use encryption systems that are based on the difficulty involved in factoring large numbers. The ever-increasing power of computers and the development of more sophisticated factoring methods are forcing cryptographers to choose even larger and more cumbersome numbers on which to base codes. On September 4, 2006, a 9,808,358-digit prime number—the largest known to date, was discovered. But on August 23, 2008, an even larger prime number was discovered—one that is 12,978,189 digits long! The search for even larger prime numbers continues.

For more information on large prime numbers, go to http://primes.utm.edu/largest.html.

This activity lesson has students using prime numbers to break a secret code. This assumes some knowledge of programming simple formulas into a computer spreadsheet. Depending upon student background with spreadsheets, you may need to spend additional time in helping students program their spreadsheets. All programming instructions for the spreadsheet (based on Excel) are included in the activity lesson.

As a "programming note," you may advise students that instead of typing the individual formulas into Cells A2 through E2, they could first type the following formula into Cell A2: =A1/$A3. Then they could use the Fill Right function to enter the formula electronically into Cells B2 through E2. The $ in the formula holds the value of a cell designation so that it does not change when you use the Fill function. Advise students that after they use the Fill Right function, they may want to click on any of Cells A2 through E2 and look at the top of the screen to see the formula that is actually in the cell. Notice that although the cell designations for the dividends increase from A1 to E1 as you go from left to right, the cell designations for the *denominators* remain A3. This is because $A3 was in the formula (instead of A3).

If students work in pairs, one student in the pair could read the prime numbers from the list, while the other enters the prime numbers into the computer. You could make a contest out of this—by challenging the class to see which pair is able to break the code first.

This activity lesson affords a "teachable moment" to "zero in" on the topic of division by 0. When students enter A1/A3 into their spreadsheet, they are confronted with the Undefined Message **#DIV/0!**—because at that moment the value assumed by the computer in Cell A3 is 0. A standard way to explain *why* we have a "commandment" stating "Thou shalt not divide by 0!" involves the use of *inverse operations* as described below.

We know that 15 ÷ 5 = 3. So, it follows that 3 × 5 = 15. (Everyone agrees on this.)

We can use the above pattern to show two cases involving division by 0:

(1) dividing a nonzero number by 0 and
(2) dividing 0 by 0.

(1) Use the above pattern to find ■: 15 ÷ 0 = ■.

Since 15 ÷ 0 = ■, then ■ × 0 = 15.

But we know that no number times 0 is equal to 15. So there is *no solution.*

A nonzero number divided by 0 is undefined.

(2) Use the above pattern to find ■: 0 ÷ 0 = ■.

Since 0 ÷ 0 = ■, then ■ × 0 = 0.

Since *any number* will satisfy ■ × 0 = 0, there is *no unique solution.*

For example, suppose you could say that 0 ÷ 0 = 3, since 3 × 0 = 0. But, you could also say 0 ÷ 0 = 4, since 4 × 0 = 0. If the above divisions were allowed, then we would have to conclude that 3 = 4, since both 3 and 4 would be equal to the same quantity (namely, 0 ÷ 0). But we know that 3 ≠ 4.

Hence, we say that 0 divided 0 by is *indeterminate* (or meaningless).

0 divided by 0 is undefined.

The above discussion generally begs the question about dividing 0 by a nonzero number, so it is often wise to include such an example to avoid possible student confusion.

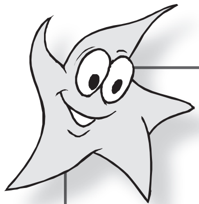
Use the above pattern to find ■: 0 ÷ 5 = ■.

Since 0 ÷ 5 = ■, then ■ × 5 = 0.

Since the product is 0, at least one of the factors must be 0. So, ■ = 0.

0 divided by a nonzero number is equal to 0.

For information on prime numbers, and in particular, very large prime numbers, go to http://www.utm.edu/research/primes/largest.html.



Mathematical Humor

Question: Why are cryptologists rarely sick?

Answer: Because they have found a cure to the common code.

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

The rule against dividing by 0 brings to mind a humorous story that makes an important point about teachers who tell students "rules" that are ultimately "broken" by other teachers in later grades. Here are some examples:

- When teaching subtraction with renaming (as in 83 – 37), a third-grade teacher may say, "You cannot subtract 7 from 3 because 7 is greater than 3. You cannot take a larger number from a smaller number." But when the student reaches sixth grade, he or she learns that with integers you can take 7 from 3. The third-grade teacher should have said, "We don't have enough ones to take 7 ones from 3 ones, so we rename 8 tens as 7 tens and 10 ones."
- When simplifying expressions such as $3a + 4b$, a teacher may explain, "You cannot combine $3a$ and $4b$ because you cannot add apples and oranges." However, later students learn that they can multiply $3a$ and $4b$ (obtaining $12ab$). So some students may ask, "Why is it that we can multiply apples and oranges, but we cannot add them?" At this point one must wonder: "Do the apples and oranges still have 'a-peel'? Did this metaphor provide a 'fruitful' experience?" Note that rather than discussing apples and oranges, it should be explained that the *distributive property* enables us to add like terms: $3a + 4a = (3 + 4)a = 7a$. However, we cannot use the *distributive property* with $3a + 4b$ because a and b are unlike terms.
- An Algebra I teacher may say, "We cannot factor $x^2 + 1$." But in Algebra II, the student learns that you can factor $x^2 + 1$ by using imaginary numbers: $(x + i)(x – i)$. The Algebra I teacher should have said, "You cannot factor $x^2 + 1$ over the real numbers."

The story picks up with a dialogue between a teacher and a sixth-grade student who has already been exposed to a few of these "broken rules."

Teacher: Division by 0 is impossible. You cannot divide by 0.

Student: Will we be able to divide by 0 when we're in seventh grade?

Solutions

Coded Message #1: Dividing each number in the code by the prime number 397 results in the following whole-number quotients: 2520, 715, 2324, 2218, and 1308. The coded message is *What's Up Doc.*

Coded Message #2: Dividing each number in the code by the prime number 757 results in the following whole-number quotients: 508, 3031, 925, 1110, 1199. The coded message is *Mrs. Peacock.*

Extensions

1. Have students encipher their own codes based on a large, common prime number, trade the codes with other students, and then try to use a spreadsheet to decipher them.
2. The following deciphering "mystery" involves an application of divisibility rules. You may want to review divisibility rules with students prior to solving the "mystery." Information on divisibility rules—and why they work—is available from the Math Forum at http://mathforum.org/k12/mathtips/division.tips.html.

Divisibility Rules for the Numbers 2–10

2 The ones digit of the number is a 0, 2, 4, 6, 8, or 10.

3 The sum of the number's digits is divisible by 3.

4 The last two digits of the number, when considered as a 2-digit number, is divisible by 4.

5 The number ends in 0 or 5.

6 The number is divisible by 2 and 3.

7 Double the final digit in the number. Subtract that result from the number formed by the remaining digits of the number. If the result is divisible by 7, the original number is divisible by 7. Otherwise, repeat the process. Example: Consider 41,923. Double the 3 to obtain 6. Subtract 6 from 4,192 to obtain 4,186. Repeat the process: Double the 6 to obtain 12. Subtract 12 from 418 to obtain 406. Repeat the process: Double the 6 to obtain 12. Subtract 12 from 40 to obtain 28. Since 28 is divisible by 7, the number 41,923 is divisible by 7.

8 The last three digits of the number, when considered as a three-digit number, is divisible by 8.

9 The sum of the number's digits is divisible by 9.

10 The ones digit of the number is 0.

The Mystery: A suspect, nicknamed "Divisibility *RULES,"* always signs his or her secret messages with a number that is divisible by 3, 5, and 7. Three recent secret messages have been found, but only one is believed to be from "Divisibility *RULES."* Below are the signatures from the three messages. Which message was likely to be from "Divisibility *RULES*"? Sorry, no calculators allowed!

296,418,357	**692,835**	***3,587,010***
signature on the first message	signature on the second message	signature on the third message

Solution: 3,587,010 is divisible by 3, 5, and 7, so the signature on the third message was likely to be from "Divisibility *RULES."*

Name____ Date ________________

Who Is the *Prime* Suspect?

Materials: computer; spreadsheet software (such as Excel)

A number is *divisible* by a second number if the quotient when dividing the first number by the second number is a whole number and the remainder is 0. A *prime number* is a positive integer (other than 1) that is divisible only by itself and 1. The first four prime numbers are 2, 3, 5, and 7. A *composite number* is a positive integer that has a positive divisor other than 1 or itself. The first four composite numbers are 4, 6, 8, and 9. The numbers 0 and 1 are neither prime nor composite.

More than 2,500 years ago, the Greek mathematician Euclid proved that there is an infinite number of prime numbers. Since prime numbers do not occur in a regular sequence—and since there is no formula to find them—mathematicians must *search* for them. Basically they use the "guzzinta" method: To check that a number is prime, *trial division* is used to determine that no number (but the number itself and 1) "goes into" that number. Today this is done with computers.

Following is the largest known prime number (as of the time of this printing), discovered in 2008:

$$2^{43,112,609} - 1$$

In the above number, the base 2 is used as a factor 43,112,609 times. To obtain the prime number, 1 then is subtracted from that result. When expanded, this number has 12,978,189 digits. If the 12,978,189 digits were written in this 12-point Helvetica typeface without commas, the numeral would extend about 20 miles. Mathematicians are continually searching for the next largest known prime number.

Very large prime numbers are used as codes to protect information stored in computers. Very large prime numbers are also used to break such codes. Because prime numbers are difficult to discover, they improve the security of computer installations. The job of a cryptologist is to encipher (create) and decipher (break) such codes.

You have just intercepted the coded message that appears below!
This message may reveal the name of the *prime* suspect in your case!

Coded Message #1

1000440 0283855 0922628 0880546 0519276

You can decode the Coded Message by following Steps 1–4 on the following pages. Each 7-digit number in the code is divisible by ONE prime number that you need to find.

So, to break the code, you must find the one prime number that is a factor of each number in the code.

You will use a spreadsheet to simultaneously divide each of the five numbers in the code by prime numbers that you test. There is one particular prime number (that you have to find) that will result in all whole-number quotients. When you simultaneously obtain all whole-number quotients, you will have found the prime number that "goes into" each number in the code. You will then apply the Decoder (on page 36) to those quotients to help you spell out the secret code.

Page 38 shows a table listing the first 200 prime numbers. That list provides the prime numbers that you need to test to break the code.

Steps to Decode the Message

1. Enter each of the five numbers from the Coded Message in row 1 of a computer spreadsheet.

	A	B	C	D	E
1	1000440	0283855	0922628	0880546	0519276

2. You will use your spreadsheet to simultaneously divide each number in row 1 by a prime number that you will enter in cell A3. To tell the computer to divide each number in row 1 by a number you enter in cell A3, type the formulas shown below into Cells A2 through E2:

	A	B	C	D	E
1	1000440	0283855	0922628	0880546	0519276
2	=A1/A3	=B1/A3	=C1/A3	=D1/A3	=E1/A3
3					

Enter each prime number you test into cell A3.

Note: A formula in a computer spreadsheet must begin with an = sign to tell the computer that an operation is about to be performed. If an = sign is not used, the computer will interpret the information as ordinary text.

Note: =A1/A3 tells the computer to divide the value in Cell A1 by the value in Cell A3. You will now get an Undefined Message (**#DIV/0!**) in Cells A2 through E2. This message is what you should expect—since you have not yet entered a nonzero divisor in cell A3. At this moment, the value in cell A3 is 0. So the computer is being asked to divide by 0—something that is impossible!

3. With a partner, use your spreadsheet to divide each of the five numbers by the same prime number until each number has a whole-number quotient. Continue to enter prime numbers in cell A3 until you find the one that works.

   Note: To increase your speed, press the *Option / Enter* (on a Mac) key after you type each prime number (rather than using the *Return* key). This will keep the cursor in cell A3. (If you use the *Return* key, the cursor will move down to the next cell.)

4. Once you obtain all whole-number quotients in Cells A2–E2, record them in the following table—with the first quotient going in the first box, the second quotient in the second box, and so on. Then use the Decoder below to write a letter for each group of two digits in your quotients. (Write leading 0s for any three-digit quotients.) The letters will form the message.

Whole-number Quotient					
Decoded Message	____ ____	____ ____	____ ____	____ ____	____ ____

DECODER for Message #1												
A 07	B 17	C 08	D 18	E 09	F 01	G 19	H 20	I 10	J 02	K 11	L 21	M 12
N 03	O 13	P 22	Q 14	R 04	S 23	T 15	U 24	V 05	W 25	X 16	Y 26	Z 06

Write out the secret message:

____

____

You have just received another Coded Message. It appears below. Obviously, the first message was useless. Use the same method and the new Decoder below to break the code. Maybe this message will reveal the name of the *prime* suspect.

Coded Message #2

384556 **2294467** **700225** **840270** **907643**

Whole-number Quotient					
Decoded Message	____ ____	____ ____	____ ____	____ ____	____ ____

DECODER for Message #2												
A 25	B 18	C 11	D 17	E 09	F 20	G 19	H 01	I 17	J 02	K 99	L 21	M 05
N 03	O 10	P 31	Q 14	R 08	S 30	T 15	U 24	V 39	W 13	X 16	Y 26	Z 28

Write out the secret message:

____

____

The First 200 Prime Numbers

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349
353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463
467	479	487	491	499	503	509	521	523	541
547	557	563	569	571	577	587	593	599	601
607	613	617	619	631	641	643	647	653	659
661	673	677	683	691	701	709	719	727	733
739	743	751	757	761	769	773	787	797	809
811	821	823	827	829	839	853	857	859	863
877	881	883	887	907	911	919	929	937	941
947	953	967	971	977	983	991	997	1009	1013
1019	1021	1031	1033	1039	1049	1051	1061	1063	1069
1087	1091	1093	1097	1103	1109	1117	1123	1129	1151
1153	1163	1171	1181	1187	1193	1201	1213	1217	1223

The Credit-Card Crunch

Teacher's Notes

■ **NCTM Standards:**	Number and Operations
■ **CCS Standards:**	Ratios and Proportional Relationships; Use appropriate tools strategically.
■ **Mathematical Topics:**	Use percents (including compound interest); make a computer spreadsheet; explore concepts related to financial literacy
■ **Grouping of Students:**	Work in small groups, in pairs, or individually

Background

A good foundation in math would improve financial literacy and help prevent younger people from making poor financial decisions that can take years to overcome.

—Alan Greenspan (1926–),
U.S. Federal Reserve Chairman (1987–2006)

Money management is one life skill that our school systems ignore all too frequently. While thousands of schools have added some form of money training to the curriculum, the truth is that students leave high school today with little or no more money savvy than previous generations.

—Charles A. Jaffe,
Financial Editor, *The Boston Globe*

The above quotes highlight the importance of including concepts related to financial literacy in the school curriculum. This activity lesson is intended to provide awareness to students of the pitfalls associated with credit-card debt. Students use a spreadsheet to explore the virtual impossibility of getting out of credit-card debt when one pays only the minimum amount due each month.

Students also explore, via the Extension, the "power of compounding." Students create a computer spreadsheet to demonstrate that when one begins saving money at an early age—such as through a retirement account—the compounding effect is spectacular. Emphasize the stark contrast between piling up credit-card debt and saving for the future.

A spreadsheet is a table of rows and columns used to help store, organize, explore, and analyze data. A computer spreadsheet is an electronic table and powerful tool used for performing a series of calculations to solve a problem. The use of computer spreadsheets in a mathematics classroom provides these instructional benefits:

- Because the computer performs the tedious and/or complex calculations, students are able to focus on the mathematics needed to solve a problem rather than on messy computations.
- Because spreadsheets show, line by line, the progression of calculations used to solve a problem, students gain insight into the powerful nature of using algorithms and formulas to solve a problem.

- Because the formulas and calculations in a spreadsheet are interrelated, students are able to “tinker” with the values of the variables and constants so that they can explore rich questions of the form “What if. . . .”
- Spreadsheets provide a real-world connection to the language of algebra. Just as variables in algebra are named with letters, so are the cells of a spreadsheet. And just as variables in algebra are replaced with numerical values, the formulas in spreadsheet cells are replaced with numbers.

In the spreadsheet that students make in question 3, point out that the formula used in column A provides an efficient way to number the months. Mention that the variables in the problem—Amount Charged, Monthly Interest Rate, and Minimum Monthly Payment—are entered into individual cells (B1, D1, and F1). By entering the variables into individual cells, they can be referred to by cell numbers in the formulas. In that way, with only minor modifications, the spreadsheet may be used to analyze problems involving different amounts, rates, and monthly payments. Note that $ signs are used in the formulas that involve D1 and F1 since those values need to remain constant when students use the Fill Down function to complete the spreadsheet.

Remind students that the Minimum Monthly Payment is $10 or 3% of the New Balance. Students will discover that beginning with Month 143, the Minimum Payment needs to be $10 (because 3% of the New Balance from that point on is less than $10). Students need to make this adjustment in their spreadsheet.

The Jump$tart Coalition of Financial Literacy

The Jump$tart Coalition of Financial Literacy (http://www.jumpstart.org/), a nonprofit organization, provides valuable resources to promote financial literacy for students. According to its homepage, “The Coalition’s direct objective is to encourage curriculum enrichment to ensure that basic personal financial management skills are attained during the K–12 educational experience.” The site also provides Web links to a number of related sites—many specifically designed for young adults—at http://www.jumpstart.org/resources.html. Additional Web sites are listed below.

Investing for Kids

http://library.thinkquest.org/3096/

This Web site is “designed by kids for kids.” It examines stocks, bonds, mutual funds, and so on. It includes a stock-market game.

Ask Dr. Math: Loans and Interest

(on the Math Forum)

http://mathforum.com/dr.math/faq/faq.interest.html

This site provides definitions and examples for key financial terms, such as simple and compound interest, Annual Percentage Rate, installment loans, The Rule of 72, and more. (The Rule of 72 provides a way to estimate how long it will take money to double at a given interest rate. According to the rule, divide 72 by the annual interest rate. For example, at 6%, your money will double in about 72 ÷ 6, or 12 years.)

Mathematical Humor

Sign outside a loan company:
Now you can borrow enough money to get completely out of debt.

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

Bank teller talking to customer whose bank account is overdrawn:
Now you know why you studied negative numbers when you were in school.

Solutions

1a. & b. Answers will vary.

2a. & b. Answers will vary.

3. Check students' spreadsheets. The final three rows of the spreadsheet are shown below.

3	Month Number	Finance Charge	Current Balance	Minimum Payment	New Balance
189	185	$0.39	$26.49	$10.00	$16.49
190	186	$0.25	$16.74	$10.00	$ 6.74
191	187	$0.10	$ 6.84	$ 6.84	$ 0.00

4a. 187 months
 b. 15.6 years

5a. $5,633.70
 b. $2,633.70

6. Answers will vary. Most students generally vastly underestimate the length of time and the total payments.

7a. People who do not pay the New Balance in full are affected by the interest rate—since they will have to pay a finance change.
 b. People who pay the New Balance in full do not pay a finance charge; hence, they are not affected by the interest rate.

8. Answers will vary. Hopefully, students will conclude that charging something for $3,000 with the idea of making just the minimum monthly payment is not a good idea. A prudent decision would be to delay the purchase until adequate funds were available.

Extension

The spreadsheet used in the Extension assumes an annual rate of return of 10%. This is because the stock market as a whole has returned an average of about 10% per year since 1928. Similar results are achieved by using slightly different rates of return. This can be demonstrated with the spreadsheet by changing the value that is entered in cell D1. The model used in the Extension assumes that each individual invests money in a tax-sheltered retirement account. Hence, for purposes of the problem, taxes on the gains are not considered.

1. The formulas that may be used to create such a spreadsheet are shown below.

	A	B	C	D	E	F	G
1			Annual Rate of Return:	0.1			
2	**Individual A**				**Individual B**		
3	Age	Deposit at Beginning of Year	Total in Account at End of year		Age	Deposit at Beginning of Year	Total in Account at at End of Year
4	22	2000	=(1+D$1)*B4		22	0	=(1+D$1)*F4
5	=A4+1	2000	=(1+D$1)*(B5+C4)		=E5+1	0	=(1+D$1)*(F5+G4)
6	=A5+1	2000	=(1+D$1)*(B6+C5)		=E6+1	0	=(1+D$1)*(F6+G5)
7	=A6+1	2000	=(1+D$1)*(B7+C6)		=E7+1	0	=(1+D$1)*(F7+G6)
8	=A7+1	2000	=(1+D$1)*(B8+C7)		=E8+1	0	=(1+D$1)*(F8+G7)
9	=A8+1	2000	=(1+D$1)*(B9+C8)		=E9+1	0	=(1+D$1)*(F9+G8)
10	=A9+1	2000	=(1+D$1)*(B10+C9)		=E10+1	0	=(1+D$1)*(F10+G9)
11	=A10+1	0	=(1+D$1)*(B11+C10)		=E11+1	2000	=(1+D$1)*(F11+G10)
12	=A11+1	0	=(1+D$1)*(B12+C11)		=E12+1	2000	=(1+D$1)*(F12+G11)

The final three rows of the spreadsheet displaying the results are shown below.

63	$0	$586,547.77		63	$2,000	$596,253.61
64	$0	$645,202.55		64	$2,000	$658,078.97
65	$0	$709,722.80		65	$2,000	$726,086.87

2. Student answers will vary. Individual A ends up with almost $710,000; Individual B ends up with a bit more than $725,000. For a long-term investment, it could be argued the ending amounts are about the same. However, Individual A invested a total of only $14,000—with no deposits after age 28. Individual B, on the other hand, invested a total of $74,000 from age 29 through age 65. Individual A made better use of the power of compounding.

Name____ Date ___________________

The Credit-Card Crunch

Materials: computer; spreadsheet software (such as *Excel*)

Many companies issue credit cards using catchy slogans to encourage people to get their cards and to buy things with them. Although ads for credit cards tend to make spending money *look like fun,* it's not much fun having to pay off credit-card debt over *many, many, many* years!

Each month your credit-card company sends you a statement showing how much you owe (the New Balance).

- If you pay the New Balance in full, *no* Finance Charges will be added to next month's bill.
- If you do *not* pay the New Balance in full, a Finance Charge will be added to next month's bill (on the old balance and on all *new* charges).

Credit-card companies use different ways to compute Finance Charges—and charge different rates. To make "it easier" to make payments, customers are generally allowed to make only a Minimum Payment each month.

Congratulations!

Your Credit Card Application Has Been Accepted!

- For unpaid balances, the Finance Charge will be 1.5% per month.
- The Minimum Monthly Payment for each billing cycle will be the greater of \$10 or 3% of the New Balance. If the New Balance is less than \$10, your Minimum Monthly Payment is your New Balance.

QUESTIONS

1. Suppose you make $3,000 worth of purchases with your credit card during the first month—and make no more future purchases.
   Quick: Make the following mental estimates (no paper and pencil):
   a. About how long do you think it would take to pay off the $3,000 charge if you make only the Minimum Payment each month?
   Estimated length of time to pay off the $3,000 charge: ____________
   b. About how much do you think your Total Payments would be to pay off the $3,000?
   Estimated Total Payments to pay off the $3,000 charge: ____________
2. Now make a few calculations and discuss the situation in your group. Then revise your estimates in question 1, if necessary.
   a. Revised estimated length of time to pay off the $3,000 charge: ____________
   b. Revised estimated Total Payments to pay off the $3,000 charge: ____________

   We are now going to make a computer spreadsheet to determine how long it will take to pay off this purchase and how much your Total Payments will be.
3. Enter the text and formulas that appear below into a computer spreadsheet. Then use the Fill Down option to fill all the rows, beginning with row 5. Notice that the Amount Charged, $3,000, the Monthly Interest Rate, 0.015 (1.5%), and the Minimum Monthly Payment, 0.03 (3%), are entered in cells B1, D1, and F1, respectively. The values in those cells are then referred to in the spreadsheet formulas.

   Note the following about the spreadsheet:
   - The Finance Charge each month is the product of the value in cell D1 (0.015) and the New Balance from the line directly above.
   - The Finance Charge each month is added to the New Balance from the line above to create the Current Balance (in cell C in the row of the given month).
   - The Minimum Payment each month is the product of the value in cell F1 (0.03) and the Current Balance (in cell C in the row of the given month).
   - The New Balance for each month is the difference between the Current Balance and the Minimum Payment during the given month.

	A	B	C	D	E	F
1	Amount Charged	$3,000	Monthly Interest Rate	0.015	Minimum Monthly Payment	0.03
2						
3	Month Number	Finance Charge	Current Balance	Minimum Payment	New Balance	
4	1	$0.00	=B4 + B1	=$F1 * C4	=C4 – D4	
5	=A4 + 1	=$D1 * E4	=E4 + B5	=$F1 * C5	=C5 – D5	
6	=A5 + 1	=$D1 * E5	=E5 + B6	=$F1 * C6	=C6 – D6	

Spreadsheet displaying the results for the first 10 months:

	A	B	C	D	E	F
1	Amount Charged	$3,000	Monthly Interest Rate	0.015	Minimum Monthly Payment	0.03
2						
3	Month Number	Finance Charge	Current Balance	Minimum Payment	New Balance	
4	1	$0.00	$3,000.00	$90.00	$2,910.00	
5	2	$43.65	$2,953.65	$88.61	$2,865.04	
6	3	$42.98	$2,908.02	$87.24	$2,820.78	
7	4	$42.31	$2,863.09	$85.89	$2,777.19	
8	5	$41.66	$2,818.85	$84.57	$2,734.29	
9	6	$41.01	$2,775.30	$83.26	$2,692.04	
10	7	$40.38	$2,732.42	$81.97	$2,650.45	
11	8	$39.76	$2,690.21	$80.71	$2,609.50	
12	9	$39.14	$2,648.64	$79.46	$2,569.18	
13	10	$38.54	$2,607.72	$78.23	$2,529.49	

4a. Based on your spreadsheet, how many months will it take to pay off the $3,000 charge? ____________

b. How many years will that be (to the nearest tenth of a year)? ____________

Use your spreadsheet to compute the total payments made during all of the months.

5a. How much will the total payments be during the time it takes to pay off the debt? ____________

b. How much of the total payments will be finance charges? (Subtract $3,000 from the total amount paid.) ____________

6. How do the results obtained from your spreadsheet compare to your original estimates and revised estimates (from questions 1–2)?

____

____

7a. Under what conditions would you be affected by the interest rate of a credit card?

____

____

b. Under what conditions would you not be affected by the interest rate of a credit card?

____

____

8. If you are considering buying something for $3,000—but don't have the money and would likely not have the money for a long time—would you charge it with the idea of making just the Minimum Payment each month? Explain what you would do.

____

____

Extension

Avoiding large amounts of credit-card debt is just one aspect of securing one's financial future. Another key aspect is saving money for the future. It is especially important to begin saving when you are young. This is because the effect of *compound interest* increases as the amount of time the money is invested increases. Compound interest is interest you receive on the amount you invest and on all previous interest received.

Let's consider two people, Individual A and Individual B. Individual A begins a retirement account by investing $2,000 each year from age 22 to 28. This person does not invest any more money into this account. Individual B begins a retirement account at age 29 by investing $2,000 each year. This person continues making a $2,000 deposit each year through age 65. Let's assume that the individuals earn 10% compound interest on their money each year. This is known as the *annual rate of return.*

1. Make and extend the computer spreadsheet shown below to determine how much money each individual will have in his or her account at age 65.

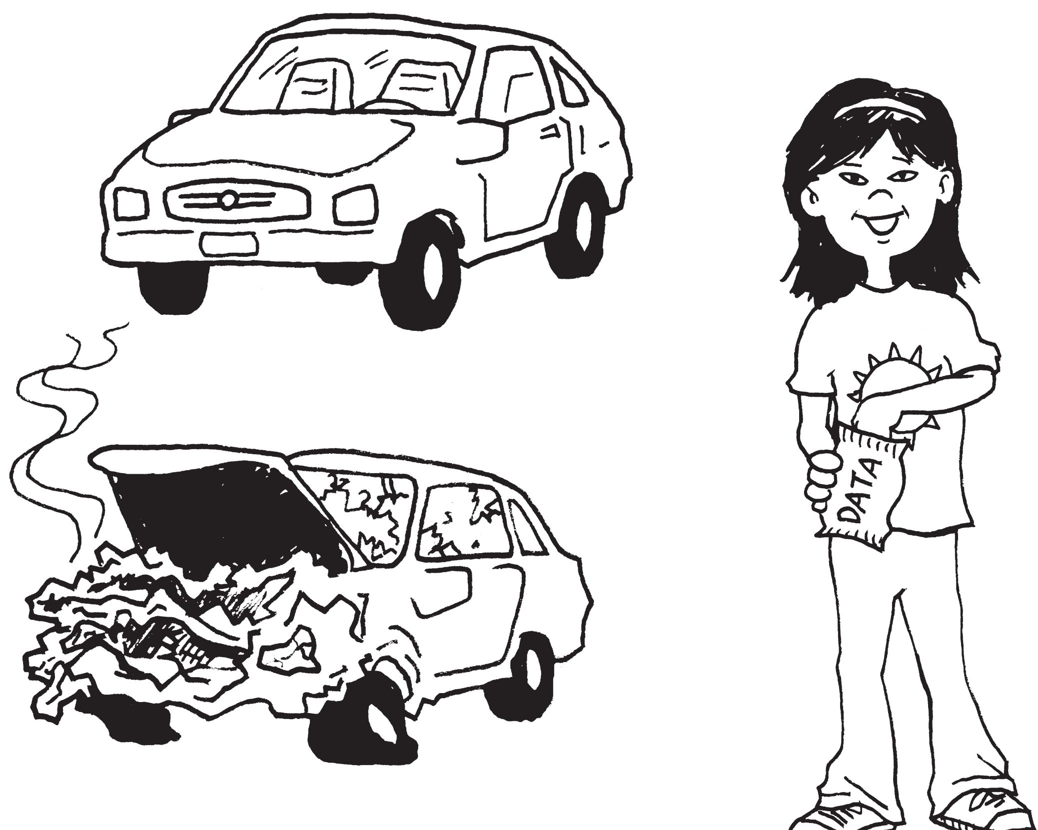
		Annual Rate of Return:	**0.10**			
Individual A				**Individual B**		
Age	Deposit at Beginning of Year	Total in Account at End of year		Age	Deposit at Beginning of Year	Total in Account at at End of Year
22	$2,000	$2,200.00		22	$0	$0.00
23	$2,000	$4,620.00		23	$0	$0.00
24	$2,000	$7,282.00		24	$0	$0.00
25	$2,000	$10,210.20		25	$0	$0.00
26	$2,000	$13,431.22		26	$0	$0.00
27	$2,000	$16,974.34		27	$0	$0.00
28	$2,000	$20,871.78		28	$0	$0.00
29	$0	$22,958.95		29	$2,000	$2,200.00
30	$0	$25,254.85		30	$2,000	$4,620.00

2. Compare and contrast the retirement plans for Individuals A and B by the time they reach age 65.

____

____

2 Focus on Data Analysis and Probability



The increased emphasis on data analysis is intended to span the grades. . . . A subject in its own right, probability is connected to other areas of mathematics, especially number and geometry.
—Principles and Standards for School Mathematics (NCTM, 2000)

Mode Code*

Teacher's Notes

- **NCTM Standards:** Data Analysis and Probability; Number and Operations
- **CCS Standards:** Statistics and Probability; Make sense of problems and persevere in solving them.
- **Mathematical Topics:** Make a frequency table; apply the concept of mode; find percents; use logical reasoning and trial-and-error
- **Grouping of Students:** Work in pairs, in small groups, or individually

Background

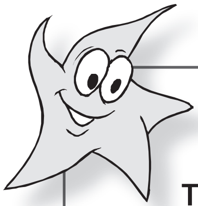
In this activity lesson, students are asked to decode a Secret Message. Their task includes making a frequency table to find the symbol in the Secret Message that occurs with the greatest frequency (the *mode*). Use the Mathematical Humor section to show situations in which a distribution may have one mode, more than one mode, or no mode at all.

The lesson works well with students working in pairs or in small, heterogeneous groups. Ideally, it is useful to have students who are strong in language arts skills (but not necessarily strong in mathematics) to be in groups with students who are strong in mathematics. Students strong in language arts often approach the task a bit differently from those who are more focused on the math. They tend to focus on word structures rather than, say, the frequency of individual letters. For example, when decoding a two- or three-letter word, they may focus more on the letter combinations that produce such a word rather than literally considering frequency of the letters in the table.

Most groups of students should be able to decode the Secret Message. If some groups have difficulty, you may want to eventually reveal a letter or two to help them move along. However, an important goal of this activity is to promote the importance of perseverance in problem solving.

You might have each group of students explain their decoding process to the rest of the class. If so, before students begin decoding the message, instruct them to have a recorder in each group take notes on how the group goes about decoding the message.

*This activity lesson is based on a "Mathematics Detective" article written by David B. Spangler for the April 1999 issue of *Mathematics Teaching in the Middle School,* copyright 1999, National Council of Teachers of Mathematics. It is reprinted here by permission of the National Council of Teachers of Mathematics. All rights reserved.



Mathematical Humor

The following can be used to further elaborate on the concept of *mode.*

- 50, 50, 50, 60, 70, 80, 90
  This distribution has one mode, 50.
- 50, 50, 60, 60, 70, 80, 90
  This distribution has two modes, 50 and 60. We say the distribution is bimodal.
- 50, 50, 60, 60, 70, 70, 80, 90
  This distribution has three modes, 50, 60, and 70. We say the distribution is trimodal.
- 50, 50, 60, 60, 70, 70, 80, 80, 90
  This distribution has four modes, 50, 60, 70, and 80. We call it *Quasimoto.*
- 50, 60, 70, 80, 90
  There is no mode because no score occurs with the greatest frequency.
  It's like the woman with three children—Eenie, Meenie, and Minie.
  What about Moe, you might ask? She didn't want "any Moe."
- And finally, there is this distribution: 3.14, 3.14, 3.14, 3.14, 90
  This distribution is called *pi ala mode.*

Solutions

1a. *E*

b. 38.5%

2a. about 165 times

It may be interesting to note that there are exactly 1,270 letters used in questions 1–6 (beginning with the word "In" and excluding any letters in the question numbers and all information given in the Frequency Table and Secret Message). The letter *E* occurs 215 times in that passage.

b. about 489 vowels

Of the 1,270 letters used in questions 1–6, there are 480 vowels (215 *E*s, 82 *A*s, 82 *O*s, 50 *I*s, and 51 *U*s).

3. See the frequency table on page 51.

4a. 17

b. 13

5. TRYING TO BREAK THIS CODE IS A REAL NEAT AND INTERESTING WAY TO EXPLORE THE CONCEPT OF MODE.

6. See the top right table on page 51. Many of the percents are close to the percents given in the table for large samples of writing. Since the Secret Message is a relatively short passage, it should not be surprising that some letters occurred more frequently or less frequently than what would have been expected in a long passage.

3.

Frequency Table			
	tallies		tallies
1	///	11	###
2	### /	12	//
3	/	13	### ////
4	//	14	///
5	/	15	###
6	//	16	//
7	/	17	### ### /
8	### /	18	///
9	/	19	### //
10	/	20	/
		21	//

6.

Numeral & Letter	Percent	Numeral & Letter	Percent
17 (E)	$\frac{11}{74} \approx 14.9\%$	16 (G)	$\frac{2}{74} \approx 2.7\%$
13 (T)	$\frac{9}{74} \approx 12.2\%$	6 (H)	$\frac{2}{74} \approx 2.7\%$
19 (O)	$\frac{7}{74} \approx 9.5\%$	21 (L)	$\frac{2}{74} \approx 2.7\%$
2 (A)	$\frac{6}{74} \approx 8.1\%$	4 (P)	$\frac{2}{74} \approx 2.7\%$
8 (N)	$\frac{6}{74} \approx 8.1\%$	12 (Y)	$\frac{2}{74} \approx 2.7\%$
11 (I)	$\frac{5}{74} \approx 6.8\%$	7 (B)	$\frac{1}{74} \approx 1.4\%$
15 (R)	$\frac{5}{74} \approx 6.8\%$	9 (F)	$\frac{1}{74} \approx 1.4\%$
14 (C)	$\frac{3}{74} \approx 4.1\%$	10 (K)	$\frac{1}{74} \approx 1.4\%$
18 (S)	$\frac{3}{74} \approx 4.1\%$	5 (M)	$\frac{1}{74} \approx 1.4\%$
1 (D)	$\frac{3}{74} \approx 4.1\%$	3 (W)	$\frac{1}{74} \approx 1.4\%$
		20 (X)	$\frac{1}{74} \approx 1.4\%$

Extensions

Have students select one or more of the following Extensions to further explore applications of cryptology:

1. Select an article from a newspaper or magazine. Make a frequency table to determine the letter frequencies for the article. Convert the frequencies to percents. Discuss how the percents for the letters in the article compare with the percents shown in the table with this activity lesson.
2. Use a book or newspaper that is written in a language other than English. Select a long passage, and make a frequency table to determine the letter frequencies for the passage. Convert the frequencies to percents to estimate the percent of use for each letter in that language. If you can write in a language other than English, you might want to write three or four paragraphs in that language on a topic of your choice—and make the frequency table based on your own writing.
3. Use an encyclopedia or other source to help you write a report on codes. Your report might deal with one or more of these areas:
   - frequency analysis
   - substitution codes
   - transposition codes
   - history of codes (including the use of codes in wars, such as the German "Enigma" in World War II)
   - use of codes to protect information, such as for pay-per-view TV signals, communication between computers, and other commercial applications
4. A standard game of SCRABBLE® comes with 100 tiles, each containing a letter and a point value. Analyze the SCRABBLE® letter frequencies—and how they relate to their respective point values and the percent each letter is used in everyday written English.
5. Interview an athletic coach to find out what kinds of codes (signals) are used in various sports.

Name____ Date ___________________

Mode Code

Cryptology is the science of secret communication. The word comes from the Greek *kryptos* ("hidden") and *logos* ("word"). People who decode messages are aided by the fact that certain letters of the alphabet occur more frequently in writing than others.

The table at the right shows the approximate percent of use of each letter in ordinary written English. The percents were calculated by counting the letters in thousands of sentences. These percents are found to be about the same when comparing one large sample of writing to another.

Work with a partner as you complete these questions.

Approximate Percent of Letters in Written English					
E	13.0%	D	4.0%	G	1.5%
T	9.0%	L	3.5%	W	1.5%
A	8.0%	C	3.0%	V	1.0%
O	8.0%	M	3.0%	J	0.5%
N	7.0%	U	3.0%	K	0.5%
I	6.5%	F	2.5%	X	0.5%
R	6.5%	P	2.0%	Q	0.3%
S	6.0%	Y	2.0%	Z	0.2%
H	5.5%	B	1.5%		

Questions

1a. In a large passage of written English, which letter would you expect to be the mode? ____________

b. In a large passage of written English, about what percent of the letters would you expect to be vowels (excluding *Y*)? ____________

2a. How many times would you expect the letter *E* to occur in a message consisting of 1,270 letters? ____________

b. How many vowels (excluding *Y*) would you expect to occur in a message consisting of 1,270 letters? ____________

3. The Secret Message on the following page was created by using numerals to replace letters of the alphabet. A given numeral in the Secret Message stands for the same letter each time it occurs. Determine the frequency for each numeral in the Secret Message. Use tally marks to record the results in the frequency table on the following page. (The Secret Message uses only 21 letters of the alphabet.)

4a. Which numeral in the Secret Message is the mode?

b. Which numeral in the Secret Message occurs with the second-greatest frequency?

5. Use the table on the previous page to help you decode the Secret Message. Assume that the numeral that occurs with the greatest frequency stands for *E*, the next-most frequent numeral stands for *T*, and so on (for many of the numerals). Since not all of the numerals in this message follow this pattern exactly, be sure to put on your detective hat—because decoding the message will require the use of trial-and-error.

6. Find the percent of use for each numeral in the Secret Message to the nearest tenth percent. Record the information in a table—from the most frequently occurring numeral to the least. Are the percents close to what you would have expected? Explain.

Frequency Table

	tallies		tallies
1.		11.	
2.		12.	
3.		13.	
4.		14.	
5.		15.	
6.		16.	
7.		17.	
8.		18.	
9.		19.	
10.		20.	
		21.	

Secret Message

13	15	12	11	8	16		13	19		7	15	17	2	10
13	6	11	18		14	19	1	17		11	18		2	
15	17	2	21		8	17	2	13		2	8	1		
11	8	13	17	15	17	18	13	11	8	16				
3	2	12		13	19		17	20	4	21	19	15	17	
13	6	17		14	19	8	14	17	4	13		19	9	
5	19	1	17 .											

What's Inside a Bag of M&Ms®?
Using Data to Make Predictions
Teacher's Notes

- **NCTM Standards:** Data Analysis and Probability; Number and Operations
- **CCS Standards:** Statistics and Probability; Attend to precision.
- **Mathematical Topics:** Make an estimate; make a frequency table; make a bar graph; calculate percents; calculate mean, range, and mode; discuss the notion of variability, use data to make predictions (inferential statistics)
- **Grouping of Students:** Work in groups of four

Background

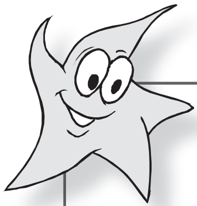
The author has used versions of this activity with students at grade levels ranging from grade K through adult learning and found that students of all ages immensely enjoy learning statistics in this manner. Students immediately take ownership in the activity by estimating how many M&Ms® are in their bag. This sets the stage to promote their curiosity as to just how many M&Ms® are in the bag—and how many of each color are in there.

The climax of the activity lesson comes at the end when students use the data they have collected and organized to make a prediction about an unopened bag (questions 11–13). Explain that statistics is the collection, organization, and interpretation of numerical data. Explain that the presentation of numerical data that describes what has happened is the branch of statistics known as *descriptive statistics.* However, the real power in using statistics is the ability to make predictions, or inferences, outside of the data that are collected—from the sample to the general population. The making of such generalizations and predictions is the branch of statistics known as *inferential statistics.* So students will be using inferential statistics when they make predictions about the unopened bag.

According to the M&Ms® Web site (http://us.mms.com/us/about/products/milkchocolate/), M&Ms® are manufactured according to the following percents of colors:

Red—13%	Brown—13%	Yellow—14%
Green—16%	Orange—20%	Blue—24%

The data appear under the heading "What Colors Come in Your Bag?" Of course, the data that students obtain in their bags will vary from bag to bag due to the variability in the processes of manufacturing M&Ms® and packaging them. You may want to compare the color percents obtained by the class to the target percents to be manufactured.



Mathematical Humor

Question: How many math teachers do you need to make a batch of chocolate-chip cookies?

Answer: Ten. One to make the batter—plus nine to peel the M&Ms®.

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

Teacher: In our ice-cream survey, 12 students voted for chocolate, 10 voted for vanilla, and 7 voted for strawberry. So the *mode* is chocolate. However, since the categories are *not* numerical, we cannot find the *mean* flavor.

Student: Yes, we can. It's Neapolitan!

Solutions

1a. Estimates will vary.

b. Some students will think that all bags contain exactly the same number of M&Ms®—and even the same number of each color. Explain that variability occurs in virtually all processes, so it is not likely that all bags contain the same number. Even airplane parts are not made with such precision. M&Ms® are likely packaged by weight, so there are going to be differences from bag to bag.

2. To check students' work, make sure the frequency table matches the graph. (Some students may actually think that you know what is inside each bag!) Also, make sure that the graph is completely labeled. And, if a student obtained a frequency of 0 for a color, make sure that the color is listed in both the frequency table and in the graph (with no squares shaded).

3. Students' frequency tables and graphs will be based on the data from their bag. Check to make sure a student's frequency table matches his or her graph. Check to make sure students' graphs are completely labeled.

4. Once the data are recorded, students should be free to "dispose" of the data.

5a. Students divide the number of M&Ms® for each color by the total number of M&Ms® in their bag.

b. The sum of the percents may be 100% or very close to 100% (due to rounding).

c. The color that occurs with the greatest frequency is the mode. If two or more colors occur with the same greatest frequency, each of them is the mode. If all colors occur with the same frequency, the mode doesn't exist (there is no mode).

Mention that the mean and the mode (along with the median—which is the middle value when a set of scores are ranked in order from least to greatest or from greatest to least) are often called measures of central tendency. Explain that the only measure of central tendency that can be used with data that are not numerical (such as colors of M&Ms®) is the mode. So, it makes sense to talk about the mode color, but not about the mean color.

6a, b, and c. Answers will vary. You might ask students if they think each group obtained the same mean and the same range. Most likely, the mean and range for each group will vary from group to group. You might also ask if students think the range for the entire class is (a) smaller than it is for their group, (b) as large as it is for their group, or (c) larger than it is for their group. Most likely, it is larger, since the range is totally dependent on the two end scores. Hence, the more data that are observed, the more likely that the two end scores will be further apart.

7. Answers will vary.

8. Answers will vary. Within each group, have students compare the modes of their individual bags. Then have them compare modes across groups. It will be interesting to see that different groups will likely have different modes.

9a. Students divide the number of M&Ms® for each color in their group by the total number of M&Ms® obtained by the group.

b. The sum of the percents may be 100% or very close to 100% (due to rounding).

10. Divide the total number of M&Ms® obtained by all students by the number of students. If each group in the class has the same number of students, you could divide the sum of the group averages by the number of groups.

11.–13. Answers will vary. In making their predictions, students should consider the mean number of M&Ms® per bag obtained by the class, the range (to take variability into account), and the percent of each color obtained by the class. Advise students that making predictions is NOT an "exact science." Rather, it is a process of making educated guesses based on data. The use of data to make predictions is likely to increase one's chances of making good predictions, but there are no guarantees. After students have made their predictions, open the bag and discuss how close the students were to what is inside the bag.

As a classroom management suggestion, you might want to handle questions 11–13 in a "game-show" environment where you are the host and the entire class participates. After students have had a chance to analyze the class data, ask the class to make a prediction in question 11 by either "calling out" their predictions (this could be quite noisy, but this tends be very exciting) or by having the class register their predictions by a show of hands. Since you are likely to obtain varying predictions, you may want to take multiple "polls" until the class settles on a single prediction. You can handle the predictions for the colors (question 12) in the same manner. However, here the students need to be careful that the sum of their color predictions is equal to their prediction for the total number of M&Ms® in the bag.

Extension

Determining the mode when data are presented in a frequency table can be confusing to some students. Present the frequency table shown below left, and ask students to determine the mode. Remind students that the mode is the item or items that occur with the greatest frequency. In that distribution, the mode is Brown (because Brown occurs with the greatest frequency). Some students, however, will erroneously conclude that "5 and 7" are the mode (because those frequencies each occur twice), or that "Yellow and Red" are the mode (because they have the greatest frequency that is repeated). If students are shown loose M&Ms® to represent this distribution, they should conclude that Brown is the mode. Ultimately, they should also conclude that a frequency table provides a compact way to analyze statistics, such as the mode.

Then present the frequency table shown below right. This is the distribution of money that is distributed to each player at the beginning of a standard game of Monopoly®. Ask students to determine the mode. The mode is $20, because more $20 bills are distributed to each player (6) than any other denomination.

Color	Frequency
Brown	10
Green	5
Yellow	7
Orange	5
Red	7
Blue	2

Bills	Frequency
$1	5
$5	5
$10	5
$20	6
$50	2
$100	2
$500	2

Name____ Date ___________________

What's Inside a Bag of M&Ms®?

Materials: 1 "Fun-Size" bag of M&Ms® per student (referred to in the activity as a "data package")

You have just been handed a "data package."

Do not tamper with the data until you are told to do so!

Before you open your data package, complete question 1.

1. a. Estimate how many M&Ms® are in your bag. __________

   b. Do you think everyone in the room has the same number in his or her bag? __________

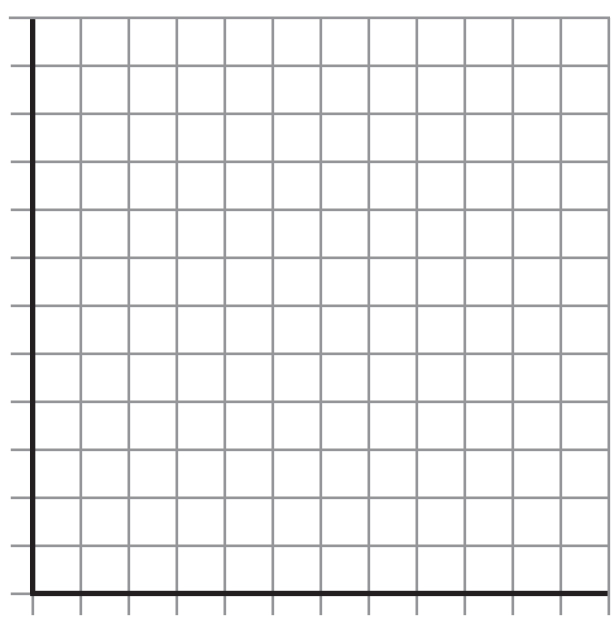
   If so, do you think each person has the same number of each color? Explain your answers.

   ____

   ____

2. Open your data package. Complete the frequency table below to show how many M&Ms® of each color are in your bag.

Color	Tallies	Frequency
Red		
Brown		
Yellow		
Orange		
Green		
Blue		



3. Use your data to make a bar graph in this grid. Make sure to label both axes of your graph and that you give the graph a title.

4. You may now "dispose" of your data in some appropriate manner.

5a. Find the *percent* of the total number of M&Ms® in your bag that is each color (to the nearest tenth of a percent).

Red _______________ Brown ________________ Yellow _________________

Orange ____________ Green _________________ Blue ___________________

b. What is the sum of the above percents for the six colors of M&Ms®? ____________

c. Which color (or colors) in your bag is the *mode* (the color or colors that occur with the greatest frequency)? ____________

6. Work in a small group.

a. In the table below, make a list of how many M&Ms® each member of your group obtained. Discuss how close you were to your original estimates.

Name	How Many
__________________________________	_______
__________________________________	_______
__________________________________	_______
__________________________________	_______

b. What is the *mean,* or *average,* number of M&Ms® per student in your group (total number of M&Ms® in your group divided by the number of students in your group)? ____________

c. What is the *range* for your group (difference between the greatest and least numbers of M&Ms® a student received)? ____________

7. Combine the data from your frequency table with the data obtained by the rest of your group. Make a new frequency table to show how many M&Ms® of each color were obtained by your entire group.

Color	Frequency
Red	
Brown	
Yellow	
Orange	
Green	
Blue	

8. Examine your group frequency table. Which color or colors is the mode? ____________

9a. For each color, find the *percent* of the total number of M&Ms® in your group (to the nearest tenth of a percent).

Red ________________ Brown ________________ Yellow ________________

Orange _____________ Green_________________ Blue ___________________

b. What is the sum of the above percents for the six colors of M&Ms®? ____________

10. Explain how you could find the *mean* number of M&Ms® per bag for the entire class. Then do so.

____

____

____

When the class gets together as a whole, we will discuss our findings and make some predictions.

Let's now consider another unopened "Fun-Size" bag of M&Ms®. We are going to use the data we have collected during this activity to help us *predict* . . .

(a) how many M&Ms® are in this bag and
(b) how many of each color are in the bag.

11. Use your data to predict how many M&Ms® are in the bag. ____________

12. Predict how many of each color are in the bag:

Red ________________ Brown ________________ Yellow ________________

Orange _____________ Green_________________ Blue ___________________

13. Discuss the data you considered in making your predictions.

____

____

____

Give Me a *Brake!*
Problem-based Investigation on the Safety of the *CrashSmasher* Automobile
Teacher's Notes

- **NCTM Standards:** Data Analysis and Probability; Number and Operations; Measurement
- **CCS Standards:** Statistics and Probability; Construct viable arguments and critique the reasoning of others.
- **Mathematical Topics:** Calculate summary statistics such as mean, median, mode, range, and standard deviation; make an appropriate graph based on the data; interpret measurements; use proportional reasoning; interpret measurements; make a presentation highlighting key data to support a point of view
- **Grouping of Students:** Work in small groups

Background

Statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write.

—H. G. Wells,
British novelist and journalist (1866–1946)

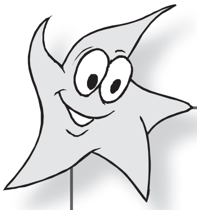
In this problem-based lesson, students analyze safety data based on a fictitious car and organize it in various ways. Students must make a key decision, based on the data, as to whether or not the car is safe for use.

To help students establish the cut-off point for a safe stopping distance when traveling at 55 mi/h, you may want to create the table (above right) of values with them. A safe driving distance is one car length for every 10 mi/h the car is traveling. The *CrashSmasher* measures 5 meters in length.

mi/h	meters	car lengths
10	5	1
20	10	2
30	15	3
40	20	4
50	25	5
55	**27.5**	**5.5**

So, at 55 mi/h, a car needs to stop within 27.5 meters.

Students may discuss additional safety factors that should be considered. For example, it's likely that cars that don't function properly at 55 mi/h will do even worse at greater speeds. Also, many drivers do not maintain safe car-length distances. The possibility that this vehicle could cause chain-reaction accidents is yet another consideration.



Mathematical Humor

Comment from an engineer, while examining a bridge designed by him that has just collapsed:

"According to my calculations, the measurements on my blueprints were only off by one decimal place. Don't you think I deserve at least partial credit for my work?"

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

If a person puts one arm into an oven, and the other arm into a freezer, some statisticians would say that *on average* the person is comfortable.

Solutions

Students need to submit a paragraph that includes relevant summary statistics and a graph. The report must also include a conclusion as to whether production of the car should cease.

Based on the data, 2 of the 40 cars, or 5%, failed the stopping-distance test. This means at 55 mi/h, 5% of the cars do not stop within the standards established by the U.S. Highway Department.

The median and mode stopping distance for the cars is 24 meters; the mean is 23.85 meters. The range is 30 – 17, or 13 meters. The standard deviation is about 2.1.

Students are likely to make a bar graph. Some will make a line plot, circle graph, or box-and-whisker plot. A line graph would not be appropriate since the data are discrete. A typical graph that students may make appears at the top of page 63.

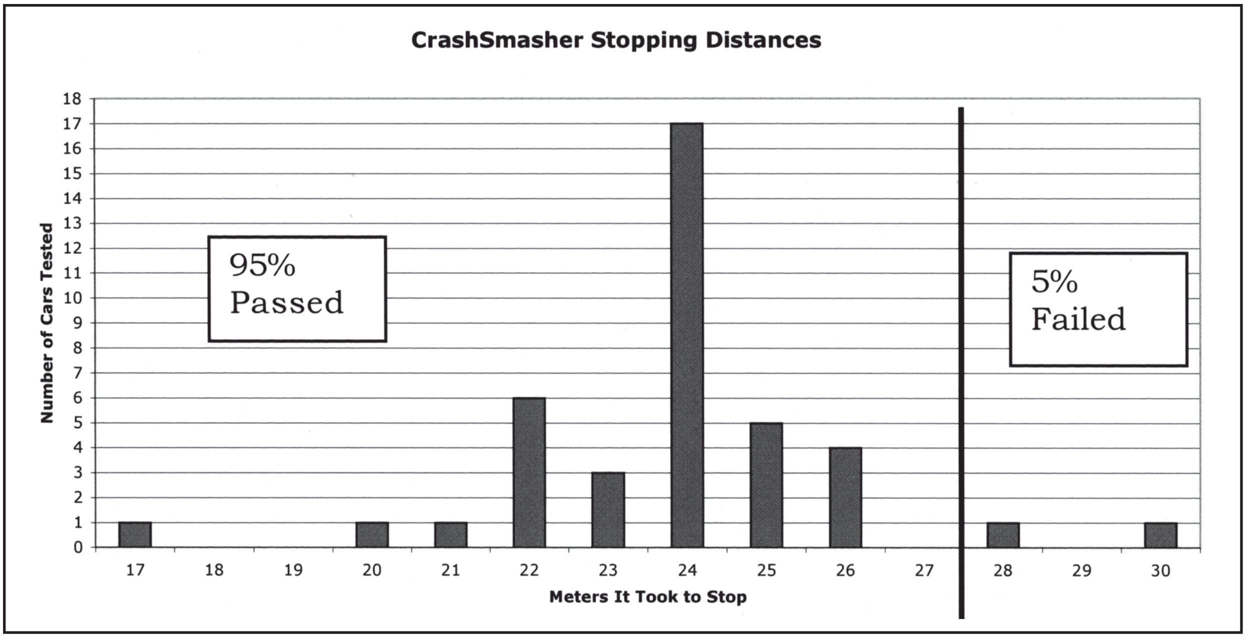
Some students may conclude that 5% is a reasonably "safe" percent and will recommend that no action be taken against the company. Advise these students that 5% is a huge percent for this situation. For example, if 100,000 *Crash-Smasher* cars were sold, 5,000 of them would be defective in their ability to safely come to a halt. (See Extension.)

Some students may "smooth out the data" by using the mean stopping distance for the 40 cars (24 meters) to suggest that the vehicle is safe. They may even conclude that the mean stopping distance provides about 13% more stopping room than the U.S.H.D. recommendation ($27.5 - 24 = 3.5$; $3.5 \div 27.5 \approx 0.13$). Advise students that when it comes to life-and-death situations, one should not use averages to "hide" or mask values that may suggest a safety problem.

Some students may make the mistake of concluding that the greater the stopping distance, the better the brakes. These students may erroneously conclude that 95% of the cars failed the test (because they are reading the data "backward"). Advise these students that cars at 55 mi/h that are able to stop within 27.5 meters perform better than those that require a greater distance.

In scoring the report/presentation, you might want to use a rubric such as the following:

- Accuracy of the summary statistics used in the report: 0–3 points
- Accuracy/appropriateness/neatness of the graph: 0–3 points
- Overall effectiveness of the narrative in the report and the oral presentation: 0–3 points



Extension

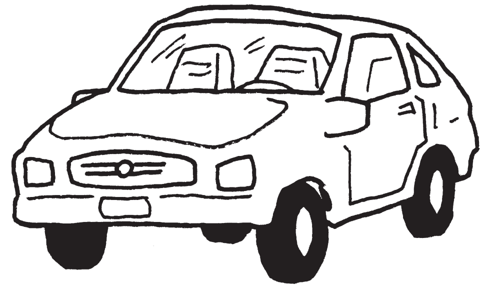
Have students research the population figures for their community, state, and for the United States. Then have them compute 5% of each of those populations. Ask if the computed numbers represent acceptable numbers of people traveling in unsafe cars. Students should conclude that although 5% may seem like a small number, the actual number of people affected in a large population is huge.

Name____ Date ____________________

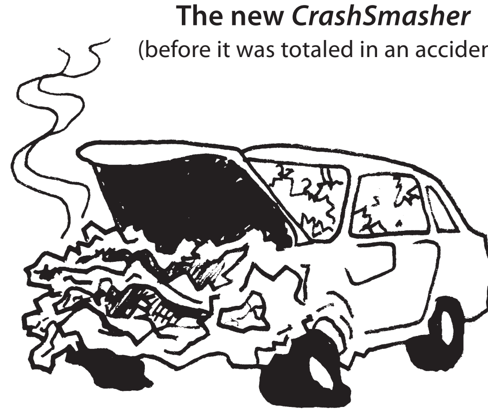
Give Me a *Brake!*

You are a member of a team of independent test engineers investigating the safety record of the *CrashSmasher* automobile. To date, thousands of *CrashSmasher* automobiles have been manufactured, and the company has great expectations for its future. However, recently there have been numerous serious accidents involving this automobile. There have been reports that these accidents occurred because the brakes on some of these cars did not function properly.

Your team's assignment is to analyze the data and then make a report that will recommend whether or not this car is safe for use on U.S. highways.



The new *CrashSmasher*
(before it was totaled in an accident)



The new *CrashSmasher*
(a "smashing hit" after a smash-crash test)

Your team of engineers has tested the brakes of 40 *CrashSmasher* vehicles. Each car was tested at a speed of 55 mi/h. Then the brakes were applied. At that point, the distance it took the car to come to a complete stop was measured. The following test results, in meters, were recorded for the 40 cars.

24	**22**	**24**	**24**	**24**	**26**	**22**	**25**
23	**22**	**24**	**24**	**20**	**25**	**26**	**30**
24	**24**	**22**	**24**	**26**	**24**	**25**	**21**
23	**17**	**25**	**24**	**24**	**26**	**24**	**22**
25	**28**	**24**	**24**	**22**	**24**	**24**	**23**

It has been determined by U.S. Highway Department standards that a safe driving distance is one car length for every 10 mi/h a car is traveling. The *CrashSmasher* measures 5 meters in length.

Group Report and Presentation

Your team of independent engineers has been asked to create a short written report on the braking system. The written report should be about one paragraph in length and include relevant *summary statistics* (such as the mean, median, mode). Most important, the report needs to provide a conclusion, indicating whether or not your team feels the car is safe for use on U.S. highways and explain why or why not. This conclusion must be clearly indicated in the report. The report must also include a *graph* (of your choice) displaying data to support your conclusion.

Your group will be expected to make an oral presentation of the report (and graph) to an important highway safety commission (your class). In your presentation, be sure to discuss what factors you considered in your conclusion. Since the safety commission is likely to ask tough questions, the style of your report and presentation should be *persuasive.*

Mozart Math:
Exploring Probability Concepts through a Musical Game

Teacher's Notes

■	**NCTM Standards:**	Data Analysis and Probability; Number and Operations
■	**CCS Standards:**	Statistics and Probability; Model with mathematics.
■	**Mathematical Topics:**	Introduction to basic concepts of theoretical probability (outcomes, equally likely outcomes, events, most likely, least likely); compare experimental data to theoretical probabilities; add fractions
■	**Grouping of Students:**	Work in small groups or in pairs

Background

Music is a hidden exercise in arithmetic, of a mind unconscious of dealing in numbers.

—Gottfried Wilhelm von Leibniz,
German mathematician (1646–1716)

This activity lesson opens with an introduction to basic concepts of theoretical probability. Students explore the possible *outcomes* when a pair of dice is rolled once, they learn about *equally likely outcomes,* and they examine collections of outcomes called *events.* Students learn how to count the possible outcomes for an event to generate probabilities for those events.

Students then move on to "Mozart's Game of Chance." Each of 11 measures of music composed by Mozart is assigned one of the dice events (2–12). Before students roll the dice, they are asked to analyze which measures of music are *most likely* to be rolled and which ones are *least likely.*

Students roll the dice, in a game akin to one played by Mozart, to create new pieces of music based on the sequence of measures that are rolled. Students then explore to see if what they actually obtain is close to what the theoretical probabilities suggest.

You may want to show students how to use *mathematical expectation* to determine how frequently each event is *expected* to occur during 12 rolls. Students could then compare these expected values to what actually occurs.

There are two common methods used to compute mathematical expectation: (1) multiplying the probability by the number of trials and (2) using proportions. Have students use one or both of these methods to compute how frequently they would expect each of the numbers 2–12 to be rolled when a pair of dice is rolled 12 times as shown on page 67.

Fractions: To predict how many times, say, an 8 is expected to come up in 12 rolls, multiply its probability by the number of rolls:

$\frac{5}{36} \times 12 = \frac{5}{3}$, or $1\frac{2}{3}$ times

Proportions: To predict how many times, say, an 8 is expected to come up in 12 rolls, set up and solve the following proportion: 5 is to 36 as x is to 12. (Based on its probability, you expect an 8 to come up 5 times in 36 rolls, so we set that ratio equal to x times 12 rolls.)

$\frac{5}{36} = \frac{x}{12}$

So, $60 = 36x$; $x = \frac{60}{36}$; $x = 1\frac{2}{3}$

So, you are expected to roll an 8 about 2 times in 12 rolls.

Mathematical Humor

Question: What do you call mathematical procedures set to music by a former U.S. Vice President?

Answer: Al-Gore-rhythm (algorithm)

Solutions

1. rolling a 5: 4 outcomes; rolling a 6: 5 outcomes; rolling a 7: 6 outcomes; rolling an 8: 5 outcomes; rolling a 9: 4 outcomes; rolling a 10: 3 outcomes; rolling an 11: 2 outcomes; rolling a 12: 1 outcome

2. Teachers should use their discretion as to whether or not they expect students to name the fractions in simplest form. For probabilities, especially those involving results based on rolling a pair of number cubes, many teachers prefer that students do not rename the fractions in simplest form. In that way, they are often better able to compare them and compute with them.

$P(2) = \frac{1}{36}$ $P(3) = \frac{2}{36}$, or $\frac{1}{18}$ $P(4) = \frac{3}{36}$, or $\frac{1}{12}$ $P(5) = \frac{4}{36}$, or $\frac{1}{9}$

$P(6) = \frac{5}{36}$ $P(7) = \frac{6}{36}$, or $\frac{1}{6}$ $P(8) = \frac{5}{36}$ $P(9) = \frac{4}{36}$, or $\frac{1}{9}$

$P(10) = \frac{3}{36}$, or $\frac{1}{12}$ $P(11) = \frac{2}{36}$, or $\frac{1}{18}$ $P(12) = \frac{1}{36}$

3a. Measure 7

b. Measures 2 and 12

4. Answers are based on what students roll.

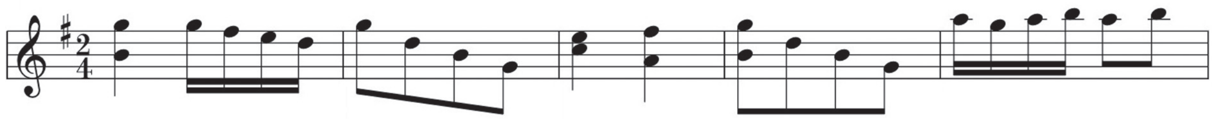
5. Answers will vary. Explanations should generally compare the frequency of those events that occurred most often and least often with those that are expected to occur most often (6, 7, and 8) and least often (2, 3, 11, and 12), respectively.

6. Students should conclude that a probability of $\frac{1}{6}$ for rolling a 7 does *not* guarantee that they will roll one 7 in every 6 rolls. Rather, a probability suggests what you are *likely to obtain* (in the long run). Students should reason that if a probability of $\frac{1}{6}$ for rolling a 7 were to mean that you are guaranteed to roll one 7 in very six rolls, then each student performing the experiment would have rolled two 7s.

7. Enjoy the music!

Extension

In the Extension, students are asked to draw vertical lines in the music, shown below, to indicate the measures. Students also are asked to show that the sum of the fractions for the notes in each measure is $\frac{2}{4}$, or $\frac{1}{2}$.



Measure 1: $\frac{1}{4} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{8}{16}$, or $\frac{1}{2}$;

Measure 3: $\frac{1}{4} + \frac{1}{4} = \frac{2}{4}$, or $\frac{1}{2}$;

Measure 2: $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8}$, or $\frac{1}{2}$;

Measure 4: $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8}$, or $\frac{1}{2}$;

Measure 5: $\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8} = \frac{8}{16}$, or $\frac{1}{2}$.

The following resource provides excellent connections between mathematics and music: *Functional Melodies: Finding Mathematical Relationships in Music* by Scott Beall, published by Key Curriculum Press (2000).

Name____ Date ____________________

Mozart Math

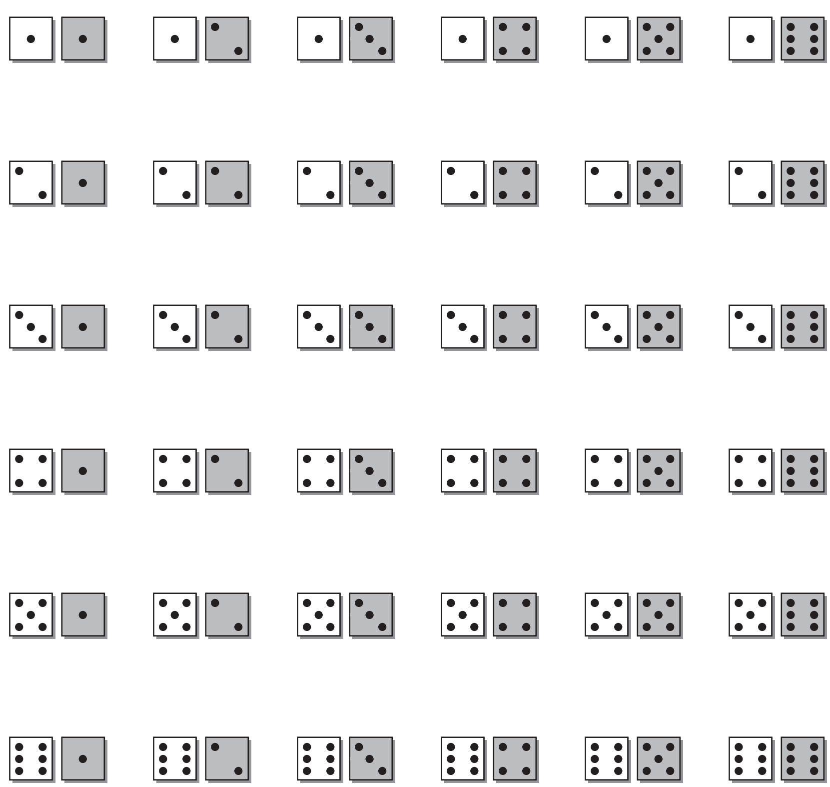
Materials: number cubes, musical instruments (optional)

Wolfgang Amadeus Mozart (1756–1791) was born in Austria and began composing when he was 5. He played before the Austrian empress when he was 6, composed his first three symphonies when he was 8, and finished his first opera at 12. Mozart lived just 35 years, but he was able to compose about 600 works.

Wolfgang Amadeus Mozart

An Introduction to Probability

Mozart combined math and music in a fun way. Before we explore those connections, let's explore a few *probability* concepts. The diagram below shows that when a pair of number cubes is rolled once, there are 36 possible *outcomes.* Since each outcome has the same *chance* for occurring, we say that each of the 36 outcomes is *equally likely.*



The diagram shows that when a pair of number cubes is rolled once, the possible sums are:

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

A collection of possible outcomes is called an *event.* Each of the above sums is an event. The diagram shows that there . . .

is 1 possible outcome for the event 2 (rolling a 1–1),
are 2 possible outcomes for the event 3 (rolling a 2–1 or 1–2),
are 3 possible outcomes for the event 4 (rolling a 3–1, 2–2, or 1–3), and so on.

Questions

1. Suppose a pair of number cubes is rolled once. How many outcomes are there for rolling the event . . .

5? _______	6? _______	7? _______	8? _______
9? _______	10? _______	11? _______	12? _______

The *probability* (or *chance*) that an event will occur can be found by writing this fraction:

$$\frac{\textbf{number of possible outcomes for event}}{\textbf{total number of equally likely outcomes}}$$

Thus, when a pair of number cubes is rolled once, the probability of rolling a 5 is

$$\frac{4}{36} \quad \frac{\leftarrow \textbf{possible outcomes for event}}{\leftarrow \textbf{total number of outcomes}}.$$

2. Suppose a pair of number cubes is rolled once. Use the diagram to help you find the probability of each of these events.

2 _______	3 _______	4 _______	5 _______
6 _______	7 _______	8 _______	9 _______
10 _______	11 _______	12 _______	

Mozart's Game of Chance

Mozart made up a game to see what happens when music is put to chance. First he wrote some measures of music. Each measure was given a number.

A person then rolled some number cubes. The total number rolled was matched to the measure of music with that number. This became the first measure of music in a new piece. Then the number cubes were rolled again. The second number rolled determined the second measure of music, and so on. When completed, Mozart and his friends had fun playing the new piece.

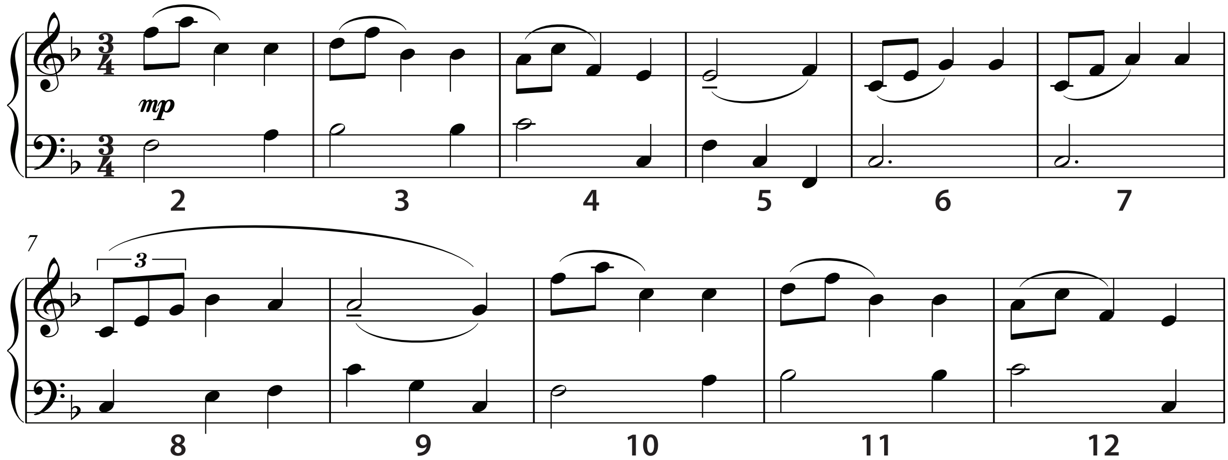
Mozart composed the music on page 71, the first 11 measures of *Minuet in F (K.2),* when he was just 5 years old. Let's use these measures of music to play Mozart's game. The 11 measures are numbered 2–12 to correspond to the numbers on number cubes.

3. Suppose a pair of number cubes is rolled once.
   a. Which measure of music below is most likely to be rolled? ____________
   b. Which two measures of music are least likely to be rolled? ____________
4. Roll a pair of number cubes 12 times. Record the sums that you roll below.

_____ _____ _____ _____ _____ _____

_____ _____ _____ _____ _____ _____

Cut out and glue the measures of music that you rolled on a piece of paper. Extra copies of music are provided at the end of this lesson since you will roll some measures more than once.



We're now going to analyze the measures of music that you rolled to create your newly created piece of music.

5. Did the measures of music you thought would be rolled more often than other measures actually get rolled more often? Did the ones with the least chance of being rolled actually get rolled least often? Explain.

____

____

____

6. Recall that when a pair of number cubes is rolled once, the probability of rolling a 7 is $\frac{1}{6}$. Based on what you discovered from this activity lesson, what does such a probability mean? Are you guaranteed to roll one 7 in every six rolls? Explain.

____

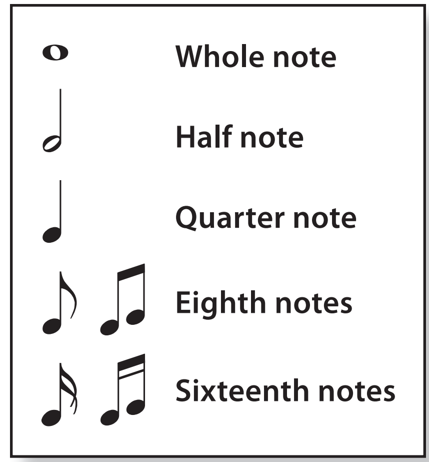
____

____

7. If musical instruments are available, members of the class may want to play the newly created pieces of music.

Extension

A Note or Two about Music: The most common types of musical notes are shown at the right. A *time signature* appears at the beginning of a piece and looks like a fraction. It tells the number of beats in a measure and type of note that receives 1 beat. For example, a signature of $\frac{2}{4}$ (two-four time) means there are 2 beats per measure and that a quarter note gets 1 beat. So, the sum of all note values in a measure of two-four time is $\frac{2}{4}$, or $\frac{1}{2}$.

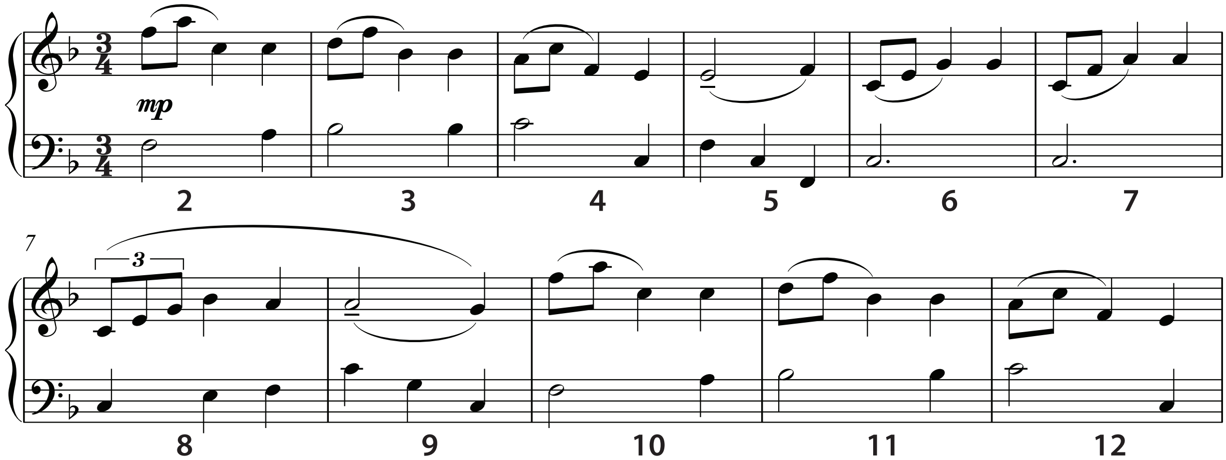


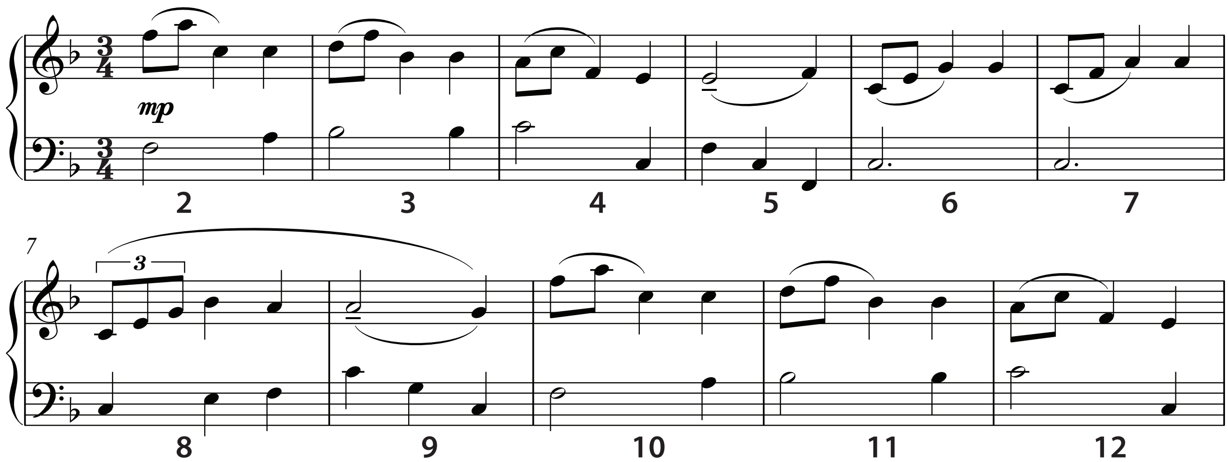
The music below, *Kontretanz,* was written by Mozart when he was about 8 years old. Draw vertical lines in the music below to separate the music into measures. Show the sum of each set of fractions below each measure. The first measure is done for you.

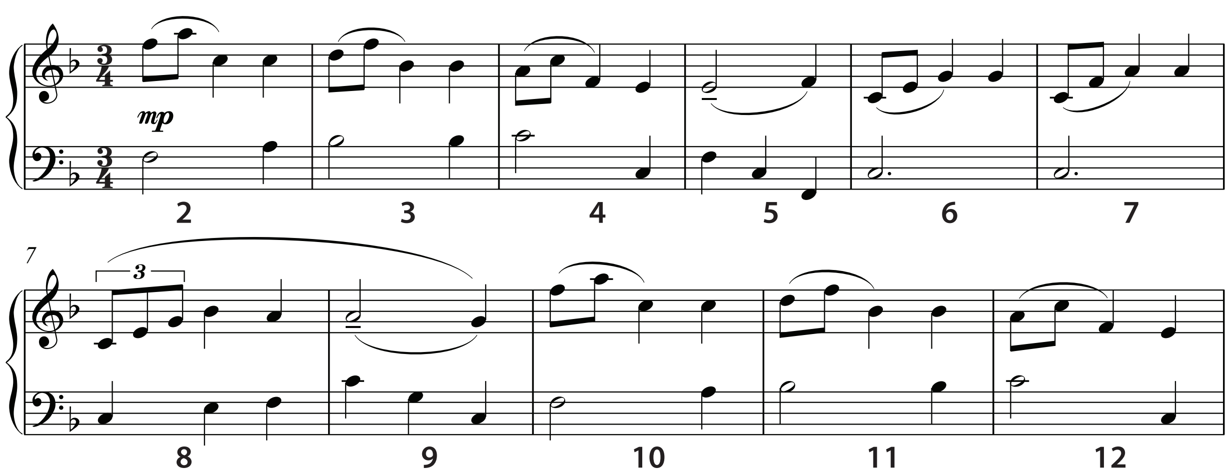
$\frac{1}{4} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{2}$

Extra Copies of Minuet in F (K.2)

Wolfgang Amadeus Mozart







Sheet music © Jim Paterson from www.mfiles.co.uk

The Case of the *Smash* Hit*

Teacher's Notes

- **NCTM Standards:** Measurement; Geometry; Probability; Number and Operations; Reasoning
- **CCS Standards:** Measurement and Data; Ratios and Proportional Relationships; Statistics and Probability; Attend to precision.
- **Mathematical Topics:** Measure to the nearest millimeter; use proportional reasoning to convert measurements based on a scale drawing; find the area of a rectangle; compute geometric and compound probability; use logical reasoning
- **Grouping of Students:** Work in small groups or individually

Background

It is a capital mistake to theorize in advance of the facts.

—Sherlock Holmes in *The Adventure of the Reigate Squire* by Sir Arthur Conan Doyle (1859–1930)

This activity lesson integrates mathematics topics from the Data Analysis and Probability, Measurement, Geometry, and Number and Operations strands in a baseball/detective setting. Further, students must use decision-making skills to explain which door is the best one to buy and then ultimately solve a logic problem in order to solve the case. The activity lesson is amply sprinkled with humor to provide additional motivation.

The type of probability applied in this lesson is often referred to as *geometric probability,* defined as follows:

If all points occur randomly in a region, then the geometric probability of an event is given as follows:

$$P = \frac{\text{measure of region for event}}{\text{measure of entire region}}$$

Students are asked to use a metric ruler to measure diagrams of storm doors and their rectangular windows. For each door, students compute the area of the glass region of the door, the area of the door, and the probability that a batted ball that hits the door at a random point will hit the glass. Students also use concepts of scale drawings to find the actual dimensions of the doors. In the end, a batted ball breaks the glass of a storm door, and Detective Shirley Solvit uses logical reasoning to determine who hit the batted ball.

Because of the deep integration of topics, this activity lesson may be successfully used with students at various points in the curriculum. This activity lesson may augment lessons on measurement, area, ratios and proportions (scale drawings), fractions, decimals, percent, and probability.

*This activity lesson is based on a "Mathematics Detective" article written by David B. Spangler for the March–April 1998 issue of *Mathematics Teaching in the Middle School,* copyright 1998, National Council of Teachers of Mathematics. It is reprinted here by permission of the National Council of Teachers of Mathematics. All rights reserved.

Mathematical Humor

As a Little-League baseball coach, the author regularly injected mathematics when attempting to give his players advice. After one game in which his 12-year-old players made numerous mental errors, he quoted Yogi Berra, a member of the Baseball Hall of Fame and humorist. According to Berra, "Baseball is 90% mental, and the other half is physical." The following week, when only eight players showed up for a game, the author gave the following advice: "90% of success in life is showing up." This caused one player—perhaps the only one who listened to anything the coach said—to comment: "Last week you said 90% is mental. Which is it?"

Whereas Yogi Berra deliberately used mathematics in a humorous way, the same cannot be said for other players. For example, during the early 2000s, there were reports that 50% of Major League baseball players were using steroids. One player pooh-poohed the reports, saying, "Well, I'm not, so that's 49% right there."

Sometimes even statesmen use mathematics in a humorous way. Former U.S. Secretary of State Henry Kissinger once said, "90% of politicians give the other 10% a bad reputation."

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

Reporter: "Miss Monroe, are you a model?"

Marilyn Monroe: "No, I am full-scale."

Solutions

1. Answers may vary slightly due to differences in student measurements.

	a. Area of Glass Region	b. Area of Door	c. Probability
Door 1:	1.6 × 3.5, or 5.6 cm^2	2 × 5, or 10 cm^2	$\frac{5.6}{10}$, or 56%
Door 2:	1.4 × 4.5, or 6.3 cm^2	2 × 5, or 10 cm^2	$\frac{6.3}{10}$, or 63%
Door 3:	1.6 × 3.5, or 5.6 cm^2	2.3 × 5.7, or 13.11 cm^2	$\frac{5.6}{13.11}$, or about 43%
Door 4:	2(1.6 × 1.6), or 5.12 cm^2	2.3 × 5.7, or 13.11 cm^2	$\frac{5.12}{13.11}$, or about 39%

2. Answers may vary. Most students will pick either Door #3 or Door #4. Door #4 has the lowest probability, 39%. Also, if a window gets hit, it would likely be cheaper to replace one of the small square windows than it would be to replace a large window on another door. However, a strong argument could also be made for Door #3. At a "Super Sale Price" of \$129, Door #3 costs only about $\frac{129}{179}$, or about 72%, of what Door #4 costs. Also, since there really isn't a large difference between the probabilities for Doors #3 and 4, Door #3 is a very good choice for Detective Solvit.

3a. $\frac{1}{16} = \frac{2}{L}$ $\qquad$ $\frac{1}{16} = \frac{5}{w}$

$L = 32$ $\qquad$ $w = 80$ $\qquad$ Dimensions of Doors #1 and 2: 32 in. by 80 in.

b. $\frac{1}{x} = \frac{2.3}{32}$

$x \approx 14$ $\qquad$ Scale for Doors #3 and 4: 1 cm ≈ 14 in.

4a. The probability that one batted ball will not hit the window of Door #1 is 1 – 0.56, or 44%.

b. The probability that neither of two batted balls will hit the window of Door #1 is
$0.44 \times 0.44 = 0.1936$, or about 19%.

c. To find the probability that at least one of two batted balls will hit the window of Door #1, consider this: Either neither ball will hit the window or at least one of them will hit the window. ("At least one" means one or both.) Solving the following equation will yield the desired result:

P (neither ball will hit the window) + P (at least one ball will hit the window) = 1

$0.19 \quad + \quad x \quad = 1$

$x = 1 - 0.19$, or 81%

Another (but more difficult) way to solve this problem is to consider the following:

P (first ball will hit the window) × P (second ball will not hit the window) +
P (first ball will not hit the window) × P (second ball will hit the window) +
P (both balls will hit the window) =
$0.56 \times 0.44 + 0.44 \times 0.56 + 0.56 \times 0.56 = 0.2464 + 0.2464 + 0.3136 = 0.8064$, or about 81%

5. **Case I: Assume Pat broke the window.**

If Pat broke the window, then Pat lied. So there is an X in the table below to show that Pat lied. But if Pat broke the window, then both Chris and Gene also lied. So Xs are also placed in the table to show that Chris and Gene lied. But recall the clue given by the bystander. Has at least one person lied and at least one person told the truth? No, so Pat did not break the window.

	Youngsters Who Lied	Youngsters Who Told the Truth
Pat	X	
Chris	X	
Gene	X	

Case II: Assume Chris broke the window.

If Chris broke the window, then Chris lied—and so did Gene. However, if Chris broke the window, then Pat did not lie. Thus, it is possible that Chris broke the window (because at least one youngster would have lied and at least one would have told the truth). It is advisable, however, to check out Case III before concluding for certain that Chris broke the window.

	Youngsters Who Lied	Youngsters Who Told the Truth
Pat		X
Chris	X	
Gene	X	

Case III. Assume Gene broke the window.

If Gene broke the window, then all three youngsters told the truth. Since the condition that at least one youngster lied and at least one told the truth is not supported, Gene did not break the window.

	Youngsters Who Lied	Youngsters Who Told the Truth
Pat		X
Chris		X
Gene		X

Conclusion: Chris broke the window (Case II).

Extensions

1. Have students choose a door in their home or school that has a window. Ask them to find
   (a) the area of the glass region;
   (b) the area of the door; and
   (c) the geometric probability that a ball that hits a random point on the door will hit the window.
2. Many students enjoy solving logic problems. Here is one that was inspired by the Abbott & Costello routine, "Who's on first?"

   There are 9 players on a baseball team, and there are 9 positions. Each player plays exactly one position. Use the following clues to determine each player's position.

   - Bud is the pitcher, Lou is the catcher, and Lefty plays left field.
   - Who plays either first base or left field.
   - What does not play shortstop.
   - I Dunno plays first base, second base, or third base.
   - Spanky plays first base or second base.
   - Babe does not play shortstop or center field.
   - No information is available on Casey's position.

   To solve the problem, have students make a table to provide a matrix of all players and positions (see below). Have them draw Xs in the cells of the row for a player to show all the incorrect positions for that player. When they determine the correct position for a player, they should draw an O in the corresponding cell. Once they determine the correct position for a player, they should fill the remaining cells in the column for that position with Xs.

   The idea is to use the information in the problem to eliminate incorrect positions for a player while also considering the possible correct positions. By continually doing this, students should be able to determine each player's position. The information for Bud is completed in the table below.

	Position								
Player	Pitcher	Catcher	1B	2B	SS	3B	LF	CF	RF
Bud	O	x	x	x	x	x	x	x	x
Lou	x								
Lefty	x								
Who	x								
What	x								
I Dunno	x								
Spanky	x								
Babe	x								
Casey	x								

Solution:

Bud—pitcher; Lou—catcher; Lefty—LF; Who—1B; What—CF; I Dunno—3B; Spanky—2B; Babe—RF; Casey—SS.

Name____ Date ___________________

The Case of the *Smash* Hit

Materials: metric ruler

Detective Shirley Solvit is shopping for a new storm door. She knows that there are youngsters who play baseball near her backyard—and that some of them can really slam the ball. So to reduce her chances of getting a broken window, she wants to buy the door that will have the greatest chance of surviving these neighborhood sluggers. Let's pick up the conversation at the store.

Clerk: May I help you?

Solvit: Can you tell me which storm door, when hit by a baseball, is least likely to get broken? Also, I'd like to see the data that support your conclusions.

Clerk: Give me a break!

Solvit: No, I'm trying to avoid anything getting broken! The conclusions will simply *hinge* on *paneless* probability.

Clerk: Probability?

Solvit: Yes. Let me explain. Suppose a batted ball hits a random point on a storm door. The probability that the ball will hit the glass portion is found as follows:

$$P = \frac{\textbf{area of glass region}}{\textbf{area of entire door}}$$

Probability is often written as a percent.

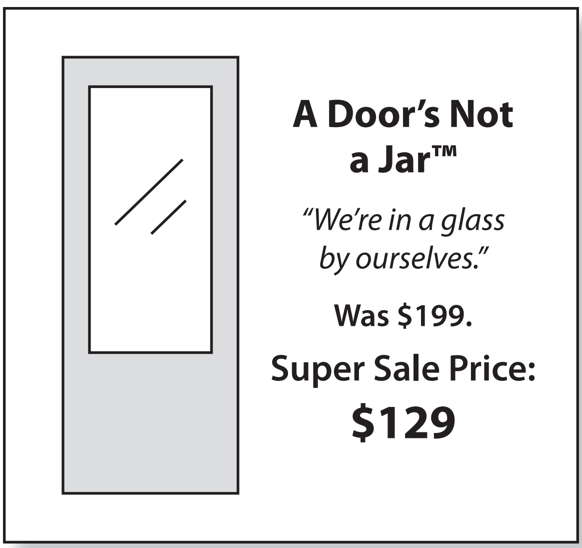
Clerk: Now that you have explained this in terms of storm doors, I can follow what you're saying. I think our readers should take some measurements of the doors, find the probabilities, and then find out which door is best for you.

The clerk and Detective Solvit have opened the door for you to swing into action to solve the following problems. You will need a metric ruler to measure the doors pictured in the following newspaper ads.

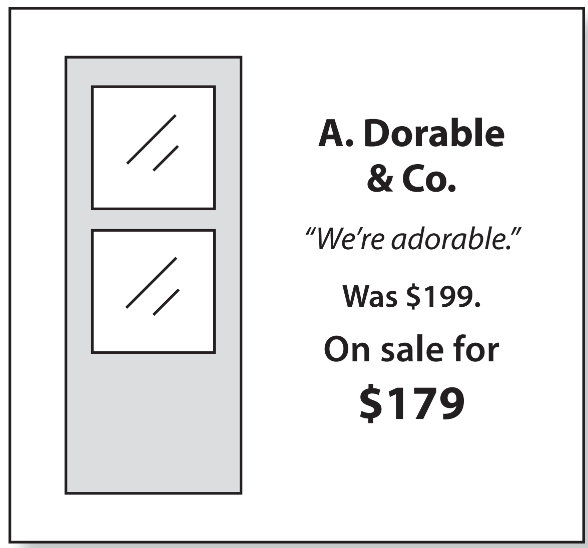
Door #1

Door #2

Door #3



Door #4



Questions

1. For each door, find the . . .
   a. area of the glass region(s). (Measure each side to the nearest millimeter.)
   b. area of the entire door.
   c. probability that a batted ball that hits the door at a random point will hit the glass. (Round each probability to the nearest percent.)

	Area of Glass Region	Area of Door	Probability
Door #1:	a. _______________	b. _______________	c. _______________
Door #2:	a. _______________	b. _______________	c. _______________
Door #3:	a. _______________	b. _______________	c. _______________
Door #4:	a. _______________	b. _______________	c. _______________

2. Detective Solvit is on a very limited budget. Which door do you think she should buy? Explain your reasoning.

____

____

3a. The scale used in the drawings of Door #1 and Door #2 is 1 cm = 16 in. What are the actual dimensions of those doors? __________

b. The actual dimensions of Door #3 and Door #4 are 32 in. by 80 in. What scale was used in the drawings of those doors? __________

4a. Suppose a batted ball hits Door #1. What is the probability that the ball will *not* hit the window? __________

b. Suppose two batted balls in a row hit Door #1. What is the probability that neither batted ball will hit the window? Round results to the nearest percent. __________

c. Suppose two batted balls in a row hit Door #1. What is the probability that *at least one* of the batted balls will hit the window? __________

Hint: To avoid getting your foot caught in the door, solve this equation:

P (neither ball will hit the window) + P (at least one ball will hit the window) = 1

Now back to the case.

Detective Shirley Solvit purchased one of the doors and installed it. Later that day, three youngsters played baseball near her backyard. All of a sudden, one of the youngsters hit a ball that went back, back, back, SMASH! It was a home run right through Detective Solvit's new storm door—and now the youngsters were wondering if they should *run home.* Detective Solvit immediately went outside and asked, "Who broke the window?"

The youngsters replied as follows:

Pat: I did not break the window.

Chris: Gene broke the window.

Gene: Chris is right. I broke the window.

Detective Solvit thought to herself, "I don't have enough information to *surely solve it.* They're playing hardball with me." Then she noticed a bystander looking on.

Solvit: Pardon me, did you see who broke the window?

Bystander: Well, I don't want to mention any names, but I'll give you this little clue: **At least one of the three youngsters has lied, and at least one of the three has told the truth.**

Detective Solvit thanked the bystander. She thought about the clue and then concluded to herself, "Now I know who broke the window. This case is open-and-shut, and no one will get framed."

5. Who broke the window? ___________

   Hint: The key to answering this question is to use *logical reasoning*— much the same way a detective uses logical reasoning. Consider these three cases:

   Case I: Pat broke the window.

   Case II: Chris broke window.

   Case III: Gene broke the window.

For each case, complete the tables provided below to help you keep track of who lied and who told the truth. Remember, based on the clue provided by the bystander, at least one person lied and at least one person told the truth.

Case I

	Youngsters Who Lied	Youngsters Who Told the Truth
Pat		
Chris		
Gene		

Case II

	Youngsters Who Lied	Youngsters Who Told the Truth
Pat		
Chris		
Gene		

Case III

	Youngsters Who Lied	Youngsters Who Told the Truth
Pat		
Chris		
Gene		

Closing comments about the case:

Before she left, Detective Solvit explained to the youngsters that she wasn't interested in throwing someone "in the slammer." "I just wanted to know who to congratulate for making that *shattering* hit," she said. As she stormed off (to go buy a new window), Detective Solvit felt good that she cracked the case of the smash hit.

Swirling Hurricane Probabilities

Teacher's Notes

- **NCTM Standards:** Data Analysis and Probability
- **CCS Standards:** Statistics and Probability; Model with mathematics.
- **Mathematical Topics:** Set up and run a simulation to estimate a probability; determine experimental probabilities; analyze data
- **Grouping of Students:** Work in pairs or individually

Background

It is remarkable, that probability, which began with the consideration of games of chance, should have become the most important object of human knowledge.

—Pierre Simon de Laplace,
French mathematician (1749–1827)

The purpose of this activity lesson is to use quantitative methods to examine how safe the residents of New Orleans were prior to Hurricane Katrina—despite assurances that they had a "200-year level of protection" from a Category 4 or 5 hurricane. In this activity lesson, students run a simulation to model what may occur (during "the long run") over a 40-year period based on a probability of "1 in 200" that a Category 4 or 5 hurricane would occur.

In mathematics, there are many problems that are of interest to students, but formal solution methods may be "beyond the scope of the class." One way to find an approximate answer to such problems is to run a *simulation.* Mathematical simulations are best used in situations in which simple theoretical methods are not available or where the solution is not what one would expect (and so a concrete handle is needed). Developing a simulation involves finding a fair *model* for the given problem. The key is that the model must have the same mathematical characteristics as the original problem. Simulations based on probability are called *Monte Carlo* simulations. (The name was chosen because of the element of chance.) The distinctive feature of the *Monte Carlo* method is the use of *random devices* such as dice, coins, spinners, or a random-number generator of a calculator or computer.

In the first part of the simulation in this activity lesson, students should discover that a probability of "1 in 200" years for a Category 4 or 5 hurricane does not necessarily mean that it will take 200 years before such a hurricane will occur. The simulation is likely to illustrate that such a hurricane could occur in less than 40 years, but it may also not occur at all.

In the second part of the simulation, students should discover that although a probability of 0.5% for a Category 4 or 5 hurricane occurring during any given year is very small, the probability that such a hurricane will strike *at least once* during a 40-year period is much greater. This can be shown using theoretical probabilities based on the following formulas for complementary events:

$$P(\text{no occurrences}) + P(\text{at least one occurrence}) = 1$$

$$\text{So, } P(\text{at least one occurrence}) = 1 - P(\text{no occurrences})$$

Theoretically, if the probability of an event occurring in any given year is 0.5%, or 0.005, then the probability that the event will not occur in any given year is 0.995. So, the probability that the event will not occur during 40 consecutive years is $(0.995)^{40}$, or about 0.82. The probability that the event will occur at least once during the 40-year period is 1 – 0.82, or about 0.18. This means that during a 40-year period, there was an 18% likelihood that New Orleans would experience at least one Category 4 or 5 hurricane. That hardly gave the area a "200-year level of protection."

The following set of instructions may be used to generate (pseudo-) random numbers using the *TI–83* or *TI–84.* Similar instructions may be used with other graphing calculators.

Pick a number to be on the *initial seed* to start the generator. Each person (or pair of students) running the simulation should use a different initial seed so that each sequence of random numbers will be different.

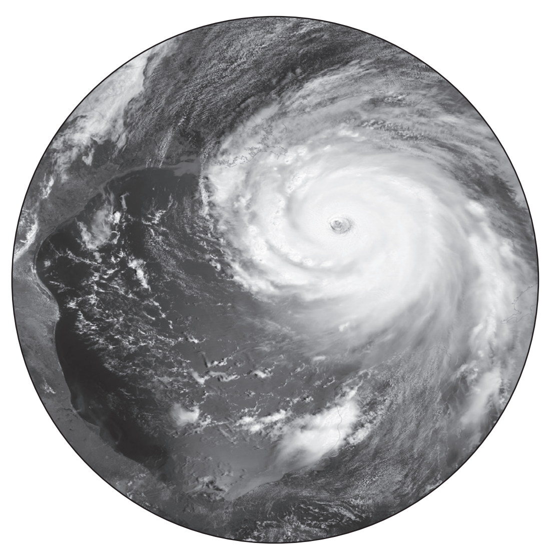
- On the Home screen, type any number (your initial seed).
- Press the $\boxed{\text{STO}\rightarrow}$ key.
- Press the $\boxed{\text{MATH}}$ key. Then scroll across to the **PRB** option at the top of the screen and press the $\boxed{\text{ENTER}}$ key to enter the **1:rand** option on the screen.
- You are now on the Home screen. Press the $\boxed{\text{ENTER}}$ key again. Your initial seed will appear at the right of the screen.

You are now ready to generate random numbers. The directions below will generate random numbers from the 1,000 whole numbers from 000 through 999.

- Press the $\boxed{\text{MATH}}$ key, scroll across to the **PRB** option, and scroll down to **5:randInt**(and press the $\boxed{\text{ENTER}}$ key.
- You are back on the Home screen. Enter the smallest number you want generated (0), then press the $\boxed{,}$ key (above the 7).
- Press the greatest number you want generated (999). Then press the $\boxed{)}$ key. So you should end with **randInt(0,999)**.
- Generate random numbers by simply pressing the $\boxed{\text{ENTER}}$ key again and again for the numbers.

Sources for this activity lesson include:

- Reason Online: http://www.reason.com/rauch/091905.shtml
- Department of Defense (including the quote from the commander of the U.S. Army Corps of Engineers): http://www.defense.gov/transcripts/transcript.aspx?transcriptid=2070



Hurricane Katrina

Mathematical Humor

Person A: The weather forecaster said that there is a 70% chance for rain this morning in our town. Just what does that mean?

Person B: It means that it will rain here during 70% of the morning.

Person C: No, it means that this morning 70% of the town will get 30% wet.

Person D: You're both wrong. It means that there's a 70% chance that the weather forecaster is 100% wet!

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

Man with family clinging to a makeshift raft, floating down a street during a huge flood: "It's a good thing there was *only* a 30% chance for rain today. Just think if it were 100%. Then where would we be?"

Note: When weather forecasters make precipitation predictions, they look at developing weather patterns together with historical records. If a particular weather pattern tends to produce rain in a given area, say, 30% of the time, then the weather forecaster may make a 30% prediction for rain. Whether it actually rains or does not rain in that area does not mean that the prediction was necessarily incorrect.

Solutions

1–3. Answers will vary. If enough trials are run (often referred to as "the long run"), simulations can provide a good estimate for theoretical probabilities. Since five whole numbers from the thousand whole numbers 000–999 are designated as Category 4 or 5 hurricanes, the experimental probability for a Category 4 or 5 hurricane occurring in any give year determined for the class in 3a is likely to be close to 0.5% (1 in 200). In 1b, however, due to the limited number of years being considered (40), a likely experimental probability is either 0 or a probability that is much greater than 0.5%. In 3c, the class experimental probability for at least one Category 4 or 5 hurricane occurring in a 40-year period is likely to be close to 20%. This is much greater than the 0.5% probability for such a hurricane occurring in any given year.

4. Answers will vary. Students are likely to conclude, depending upon their experimental probabilities associated with their 40-year models, that the levee system provided nowhere near a "200-year level of protection." The commander (and others) did not address the probability of what is likely to occur over an extended period of time—which is much greater than the probability associated with any given year.

Extension

The theoretical probabilities underlying the simulation in this activity lesson make use of the idea that it is generally much easier to calculate the probability of having a consecutive string of "no occurrences" than it is to calculate the probability of "at least one occurrence" (because finding the probability of "at least one occurrence" often involves looking at many different possible combinations. As such, to find P(at least one occurrence), we generally calculate $1 - P$(no occurrences).

A classic probability problem that also makes use of this concept is known as "The Birthday Problem." Presented below, this problem generally works best when presented to a group of, say, 25 or more people.

1. How many people would you need in this room to be *certain* that at least two people in the room share the same birthday? (366 people, based on a 365-day year)
2. What do you think is the probability that at least two people in this room share the same birthday? (The answer, of course, is dependent upon the number of people in the room. However, most people respond with a probability that is much lower than what it actually is.)
3. **The Birthday Problem:** How many people are needed in a room in order for the probability to be greater than 50% that at least two people in the room share the same birthday? (Many people give a response of about 183 people. Tell students that they are about to explore what this number actually is.)

Guide students to devise a strategy to solve the problem. You may want to suggest that to find the probability of at least one match, it is much easier to find the complement, as shown below.

$$P\text{ (at least one match)} = 1 - P\text{ (no matches)}$$

The computations and table below show how to find the probability of no (birthday) matches among various numbers of people in a room.

- Assume that we are working with a 365-day year.
- To find the probability of no matches in the room, the first person could have any birthday of the 365 days in a year. This is expressed as $\frac{365}{365}$.
- Now consider the second person in the room. There are 364 possible birthdays out of the 365 days in the year for the second person's birthday to *not* match the birthday of the first person. So we write $\frac{364}{365}$. We multiply $\frac{365}{365}$ and $\frac{364}{365}$ because we want both events to occur.
- Now consider the third person. For this person's birthday to *not* match the birthday of either the first person or the second person, there are 363 possible birthdays out of the 365 days. So we write $\frac{363}{365}$ and multiply it by the first two probabilities.
- We continue the process for each person in the room as shown below.

$$P\text{ (at least one match)} = 1 - \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \frac{361}{365} \times \frac{360}{365} \times \frac{359}{365} \times \frac{358}{365} \times \frac{357}{365} \times \ldots$$

Converting each fraction to a decimal yields the following.

$$P\text{ (at least one match)} = 1 - 1 \times .997 \times .995 \times .992 \times .989 \times .986 \times .984 \times .981 \times .978 \times \ldots$$

So, the probability of no matches when there are two people is 1 × .997, or .997; the probability of no matches when there are three people is 1 × .997 × .995, or .992; and so on. When these results are placed in a table, it is revealed that when just *23 people are in a room,* there is a .508 probability that there will be at least one birthday match. Thus, there is a greater than 50% probability that *at least two* people will share the same birthday when just 23 people are in a room!

Number in Group	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
Probability of no matches	.997	.992	.984	.973	.960	.944	.926	.906	.883	.858	.833	.805
Probability of at least one match	.003	.008	.016	.027	.040	.056	.074.	.094	.117	.142	.167	.195

Number in Group	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**. . .**	**60**
Probability of no matches	.776	.747	.716	.685	.652	.620	.588	.556	.524	**.492**	. . .	.006
Probability of at least one match	.224	.253	.284	.315	.348	.380	.412	.444	.476	**.508**	. . .	.994

The above result tends to baffle people. One way to explain this result is to ask people to think about how unlikely it would be for everyone in a room to have a different birthday. Keep in mind that any person in the room can match with any other person. Note in the table that when there are just 60 people in a room, there is a .994 chance that at least two will have the same birthday. But also keep in mind how unlikely it would be for all 60 people to each have a different birthday.

You may want to survey each student in your class to create a list of birthdays in the classroom. Then check to see if your experimental data support the theoretical probabilities.

Name____ Date ___________________

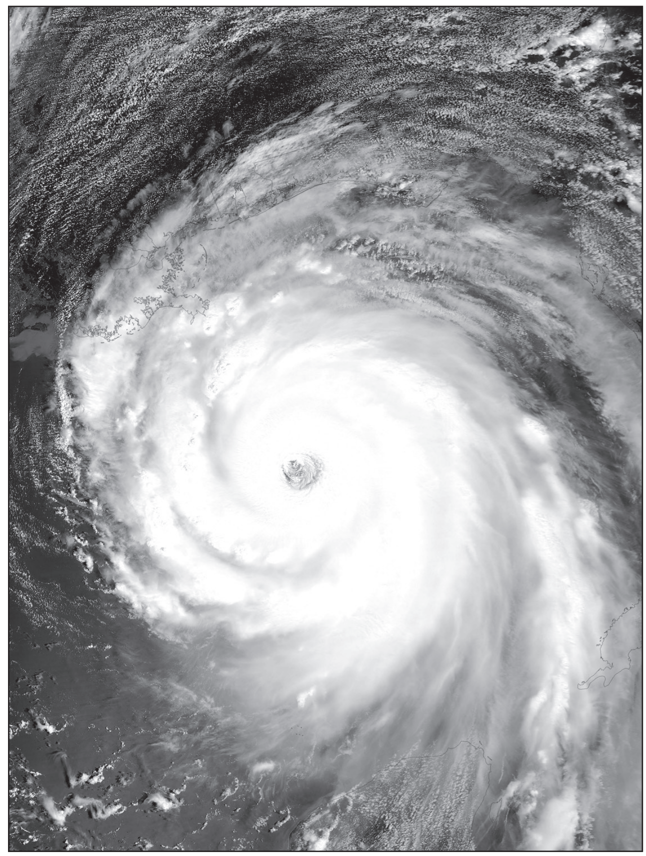
Swirling Hurricane Probabilities

Materials: calculator with a random number generator

New Orleans is built between a lake, a river, and the Gulf of Mexico—but the city is lower than the surrounding waters. It is kept dry by an extensive system of levees and pumps. The levee system, built in the 1960s, was designed to withstand a Category 3 hurricane.

Katrina hit Louisiana on August 29, 2005, as a Category 4 hurricane with sustained winds of 145 mi/h. On September 2, 2005, the commander of the U.S. Army Corps of Engineers was quoted as saying:

> *The odds of a Category 4 or 5 hurricane hitting New Orleans in any given year are small. We figured we had a 200-year level of protection. That means that an event that we were protecting from might be exceeded every 200 years. So we had an assurance of 99.5% that this would be OK. Unfortunately, we have had that 0.5% activity here.*



Hurricane Katrina

Probability Simulation

Let's run a simulation to investigate the math used in that quote. Did the 40-year-old levee system really provide a "200-year level of protection"? Suppose the occurrence of an event has a probability of 1 in 200 years (0.5% chance in any given year). Does this mean it will take 200 years before the event will occur? This simulation will generate an estimate for the probability of a Category 4 or 5 hurricane occurring during a 40-year period.

There are 1,000 whole numbers from 000 through 999. Certain calculators can be used to randomly generate numbers. Assign any five whole numbers from the interval 000–999 to the event "Category 4 or 5 hurricane occurring." (This represents a probability of 5 in 1,000—which is equivalent to 1 in 200.)

- Record your five numbers: ______ ______ ______ ______ ______
- Use a calculator to generate 40 random numbers to represent what happens during each of 40 years. Record the 40 generated numbers below.

______ ______ ______ ______ ______ ______ ______ ______

______ ______ ______ ______ ______ ______ ______ ______

______ ______ ______ ______ ______ ______ ______ ______

______ ______ ______ ______ ______ ______ ______ ______

______ ______ ______ ______ ______ ______ ______ ______

Questions

1a. How many Category 4 or 5 hurricanes occurred during your 40-year trial? ___________

b. What is your *experimental probability* of a Category 4 or 5 hurricane occurring during any given year? (Divide your result in 1a by 40.) ___________

2. Survey each classmate to find out if a Category 4 or 5 hurricane occurred during his or her 40-year trial. Write Y or N in the table below for each 40-year trial. If Yes, find out how many Category 4 or 5 hurricanes occurred, and record that number in the table. Be sure to include the data for your 40-year trial in the table.

Trial	1	2	3	4	5	6	7	8
Did a Category 4 or 5 hurricane occur during this 40-year trial? Write Y or N.								
If Yes, how many?								

Trial	9	10	11	12	13	14	15	16
Did a Category 4 or 5 hurricane occur during this 40-year trial? Write Y or N.								
If Yes, how many?								

Trial	17	18	19	20	21	22	23	24
Did a Category 4 or 5 hurricane occur during this 40-year trial? Write Y or N.								
If Yes, how many?								

Trial	25	26	27	28	29	30	31	32
Did a Category 4 or 5 hurricane occur during this 40-year trial? Write Y or N.								
If Yes, how many?								

3a. Find the class experimental probability for a Category 4 or 5 hurricane occurring in any given year.

Divide: [total number of Cat. 4 or 5 hurricanes that occurred in all trials] by [total number of trials × 40] ___________

b. Let's examine each 40-year trial that was run. In how many of those 40-year trials did at least one Category 4 or 5 hurricane occur? ___________

c. Find the class experimental probability for at least one Category 4 or 5 hurricane occurring during a 40-year period. (Divide the result in 3b by the number of 40-year trials that were run.) ___________

4. In analyzing the 40-year trials, do you believe that there really was a "200-year level of protection"? Explain.

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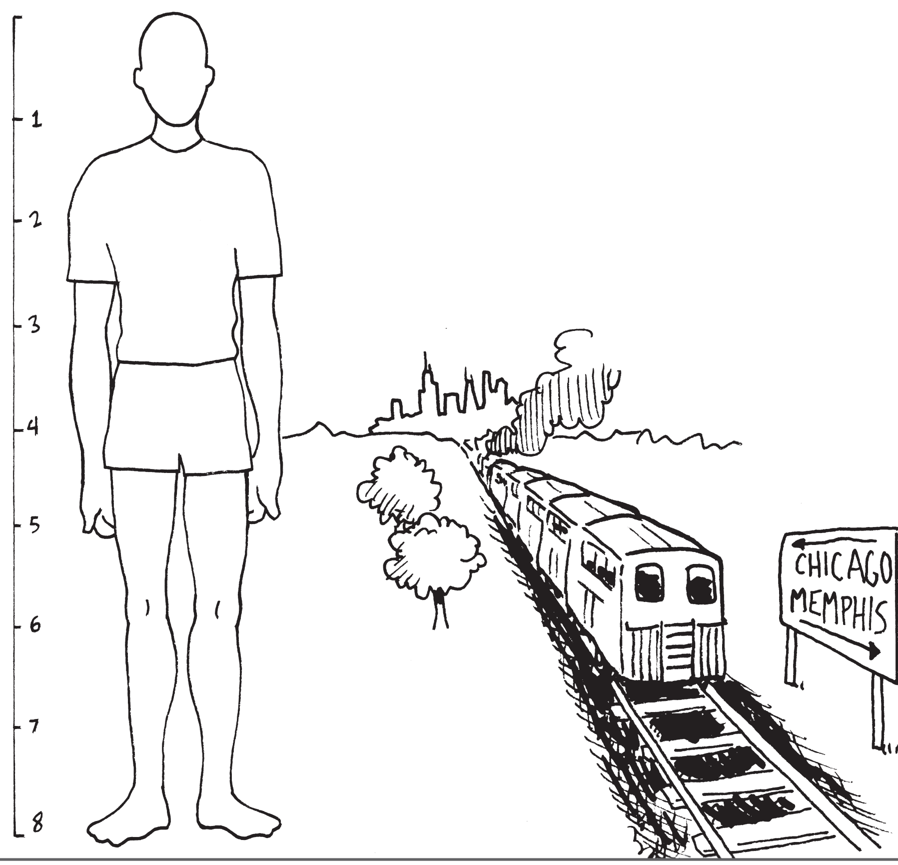
3 Focus on Measurement and Geometry



Measurement concepts and skills can be developed and used throughout the school year rather than treated exclusively as a separate unit of study. . . . Geometry provides a rich context for the development of mathematical reasoning.

—Principles and Standards for School Mathematics (NCTM, 2000)

Does Your Head *Measure Up?*
Exploring Ratios in Body Measurements
Teacher's Notes

- **NCTM Standards:** Measurement; Number and Operations
- **CCS Standards:** Measurement and Data; Ratios and Proportional Relationships; Attend to precision.
- **Mathematical Topics:** Measure to the nearest millimeter; use ratios and proportions; apply the concepts of golden rectangle and golden section
- **Grouping of Students:** Work in pairs

Background

Without mathematics, there is no art.

—Luca Pacioli (1445–1517),
Italian mathematician, artist, and
teacher of Leonardo da Vinci

In this lesson, students use their head as a non-standard unit of measurement.The body ratios used are based on the observations of sixteenth-century Italian Renaissance artist Michelangelo, who concluded that the ideal body height is "8 heads." Other sculptors, such as the Greek sculptor Praxiteles (circa 340 B.C.), also used "8 heads" as the ideal body height. It should be noted, however, that not all artists have agreed on an "8-head" model. For example, the Greek sculptor and mathematician Phidas (500 B.C.–432 B.C.) based his work on a system of "7.5 heads."

When students take the body measurements, they should use rulers extended from the various body parts, parallel to the floor. They should then take measurements between the two extended rulers. This should result in more accurate measurements (reinforcing the notion that they are finding the distance between two points), and this will keep the partner who is taking the measurements from actually touching the person being measured.

Students also explore the golden ratio and test whether their head and hand are golden. The *golden ratio* is an *irrational number* that is equal to 1.6180339887498948482 . . . (An irrational number is a number that has an infinite, non-repeating decimal expansion. Examples of irrational numbers include π and $\sqrt{2}$.) This ratio was known to the ancient Greeks as the *golden section* and to Renaissance artists as the *divine proportion.* It is also called the *golden mean.*

In the early 1900s American mathematician Mark Barr used the Greek letter phi (ϕ) to designate this ratio. Phi is generally pronounced "fy." Phi is the first letter of Phidas, who studied this ratio and applied it to the design of sculptures for the Parthenon.

There may be more than just an aesthetic value to having a "golden face." According to some studies, people with "shorter faces" (face ratio somewhat less than 1.62 to 1) tend to have more headaches than those whose faces are close to the golden ratio. People with "longer faces" (face ratio somewhat greater than 1.62 to 1) tend to have more sinus problems than with faces close to the golden ratio. For more information, go to http://goldennumber.net/health.htm.

The Extension includes a historical note about Leonardo da Vinci. Da Vinci provided illustrations for a dissertation published by Luca Pacioli in 1509 entitled "De Divina Proportione," perhaps the earliest reference to the phrase *Divine Proportion.* Da Vinci's "Vitruvian Man" was named after Marcus Vitruvius Pollio, a Roman architect in the first century who used the golden ratio in his work.

There are many outstanding Web sites that provide real-world applications of the golden ratio. One in particular is http://goldennumber.net/. This site prides itself in being "dedicated to providing you with the 'phinest' information on the golden section, ratio, or mean." Another excellent site is "Fibonacci Numbers and the Golden Section in Art, Architecture, and Music" at http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibInArt.html. It provides everything you wanted to know on the subject—from the Parthenon to da Vinci.

Mathematical Humor

In this activity lesson, students use the *head* as a basic unit of measurement. You may want to give students a *heads-up* on the following rather unusual "units of measurement"—and their related "conversion factors."

10 millipedes	=	1 centipede
1 million microphones	=	1 phone
1 trillion microphones	=	1 megaphone
2000 mockingbirds	=	2 kilomockingbirds
1,000,000 aches	=	1 megahertz (or 1000 kilohertz)
453.6 graham crackers	≈	1 pound cake
1 billion mosquitoes	≈	1 gigabyte
$\frac{1}{2}$ large intestine	=	1 semicolon
10 cards	=	1 decacards (Note that 52 cards = 5.2 decacards)
2000 lb of Chinese soup	=	won ton
Time between slipping on a peel and smacking the pavement	=	1 bananosecond

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

Question: What did the metric system do to the customary system?

Answer: It decimated it.

Solutions

1. Measurements will vary.
2. Ratios will vary. Based on Michelangelo's 8-head model, student body measurements may yield the following ratios: for a, the ratios may be about 3 to 1; for b, about 8 to 1; and for c, about 2 to 1.
3. Answers will vary. Students should address how close their ratios are to those suggested by the work of Michelangelo: Distance from the head to the waist—about 3 heads; total height—about 8 heads; ratio of total height to he length of the legs—about 2.

4a. about 1.61 (based on a length of 5.3 cm and a width of 3.3 cm)

b. Yes

5 & 6. Answers will vary.

Extension

Measurements and answers for the Extension will vary. The class ratio in part d will likely be close to 1.62.

This activity may be further extended by having students make a *scatter plot* of the individual student data. Students should graph the following ordered pairs on a coordinate grid:

distance from the floor to the student's *navel,* student's *height*

Thus, the distance from the floor to the navel is the x value; the total height is the y value.

Students should then draw a *line of best fit* through the data. This generally involves students using a ruler to draw a line through a major cluster of the data. Since the ordered pair formed by the mean of the x values and the mean of the y values is always on the line of best fit, this ordered pair should be graphed and used as a guide as the line is drawn.

Based on student background, you could also have students find the slope of the line (likely to be about 1.6) and determine the equation for the line.

Name____ Date __________________

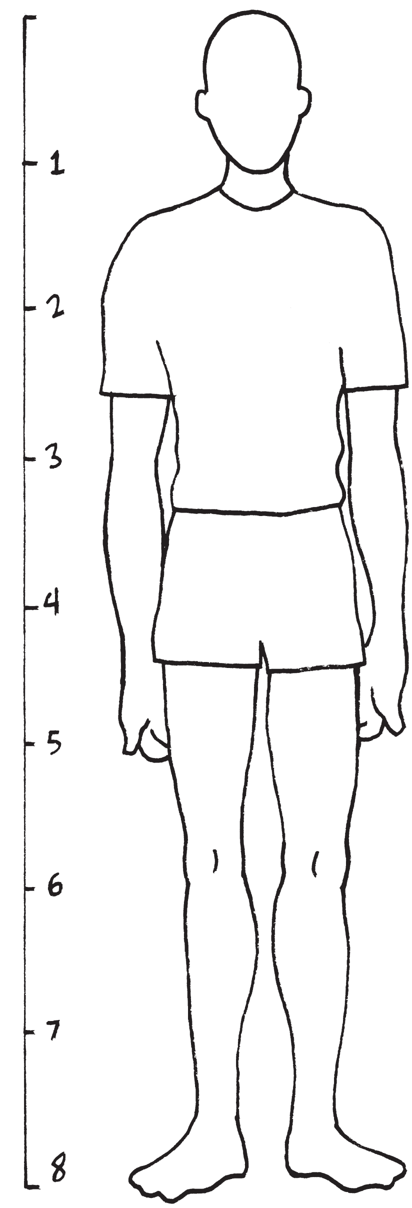
Does Your Head Measure Up?

Materials: tape measure (metric) and rulers

Over the years, many artists and sculptors have used the *head* as the basic unit in their system of body measurements. The length of the head was used to help them determine the relative size of the various parts of the human body.

The Italian artist Michelangelo (1475–1564) based the height of a human being at being about 8 “head-lengths” tall. The drawing at the right is based on the following measurements:

- The *navel* is determined at “3 heads” (counting down from the top of the head).
- The *legs* are one-half of the body length.
- The *knees* are at “6 heads.”
- The *ankles* are at “7.5 heads.”



Questions

1. Measure each of the following to the nearest millimeter. For each measurement, extend rulers from your body, parallel to the floor. Then have your partner use a tape measure to find the distance between the rulers (or between a ruler and the floor).

   a. your head length (top of head to your chin) ___________

   b. distance from top of your head to your navel ___________

   c. your total height ___________

   d. length of your legs (from your hip joint to the floor) ___________

2. A *ratio* is a comparison of two numbers by division. Find each of the following ratios of your body measurements (to the nearest tenth):

a. $\frac{\text{head to navel}}{\text{head length}} \approx$ b. $\frac{\text{total height}}{\text{head length}} \approx$ c. $\frac{\text{total height}}{\text{leg length}} \approx$

3. Describe how your ratios in Question 2 compare to the ratios suggested by the drawing on the previous page.

____

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A *golden rectangle* is a rectangle with a ratio of length to width that is about 1.62 to 1. Since ancient times, many people have believed that the most pleasing rectangle to look at is a rectangle with a length and width that are in that proportion.

4a. Measure the length and width of this rectangle to the nearest millimeter. What is the ratio of the rectangle's length to width (to the nearest hundredth)? ___________

b. Is the rectangle golden? ___________

5. Do you have a *golden face?* Use the measurement of the top of your head to your chin from question 1a. Measure the width of your head from cheek to cheek.

a. What is the ratio of the length of your head to its width? ___________

b. Is your face "golden"? (Try drawing it in the above rectangle.) ___________

6. Do you have a *golden hand?* Notice that your hand, when opened with your fingers touching each other, forms a shape that suggests a rectangle. Use the distance from the edge of your hand to the end of your middle finger as the length, and use the distance across your hand to your thumb as the width.

a. What is the ratio of your hand's length to width? ___________

b. Is your hand "golden"? ___________

Extension

The *golden section* is the division of a line segment where the ratio of the length of the entire line segment to the length of the longer section is about 1.62 to 1. Point *P* is at the golden section of line segment *AB* below because the ratio of the entire length (8) to the length of the longer section (5) is about 1.62 (8 ÷ 5 = 1.6).

The classical sculptors frequently used the navel as the "golden cut" of the body. Is your navel the "golden cut" of your body? Work with a partner to take some measurements—and find out.

a. Measure your total height. ___________

b. Measure the distance from the floor to your navel. ___________

c. Compute this ratio (to the nearest hundredth):

$$\frac{\text{your total height}}{\text{distance from the floor to your navel}} \approx$$

d. Find the average (mean) ratio for the entire class (to the nearest hundredth). ___________

e. Would you say that the navel is at the "golden cut" for the average member of your class? (To be golden, the average need not be "right on the button.")

____

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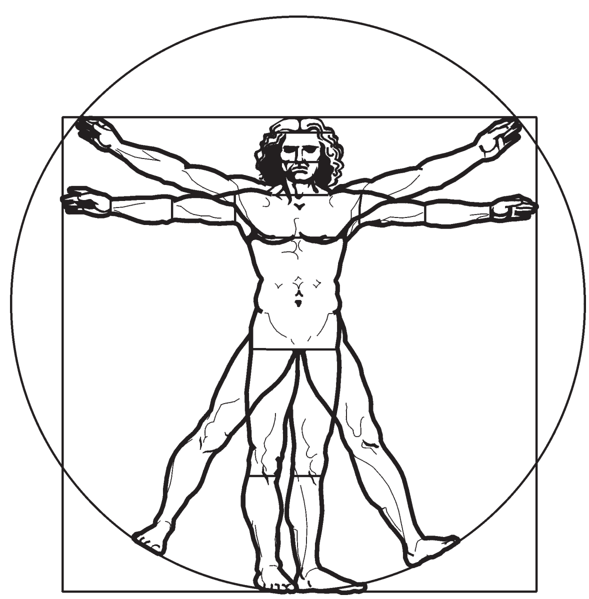
Historical Note

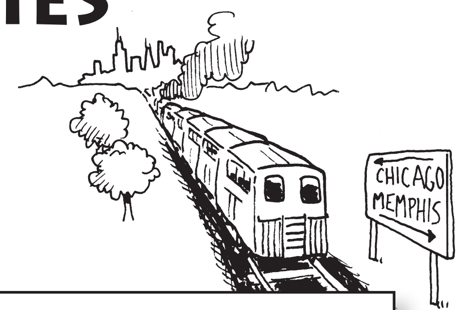
Leonardo da Vinci (1452–1519) extensively studied the proportions of the human body. Da Vinci's drawings showed the navel at the golden section of a human's height. Da Vinci also made the following observation: "If a man stands upright with hands outstretched, his height is equal to his breadth (length of the square). If he stands with arms and legs akimbo and his navel is the center, then his extremities describe a circle."



Da Vinci's "Vitruvian Man"

USING MEASUREMENT TO PUT LOTTERY PROBABILITIES INTO *PERSPECTIVE*

TEACHER'S NOTES



- **NCTM Standards:** Measurement; Probability; Number and Operations
- **CCS Standards:** Measurement and Data; Statistics and Probability; Model with mathematics.
- **Mathematical Topics:** Convert inches to miles; convert centimeters to kilometers; convert kilometers to miles; read mileage tables; understand basic probability concepts; use a linear representation to model a situation
- **Grouping of Students:** Work in pairs or individually

BACKGROUND

A lottery is a tax on people who are bad at math.

—Dana Blankenhorn,
business journalist (1955–)

The goal of this activity is to use concepts from measurement to provide a visualization for lottery probabilities. Students should conclude from their linear representations that "investing" money in a lottery (beyond, say, a token amount) is *not* a wise financial decision. This should come to light as they think about trying to pick *the* "lucky cup"—at $1 a pick—from a line of cups stretched for thousands of miles.

The probabilities provided in this activity lesson were in use as of mid-2007. Probabilities are adjusted from time to time. For the latest information on *Powerball,* go to http://www.powerball.com/. For the latest information on *Mega Millions,* go to http://www.megamillions.com/.

Students should use a calculator when they convert from inches to feet to miles. Before students use the mileage tables to determine their "itineraries," you may want them to refer to a map or globe to estimate and show how far they *think* the cups will extend—beginning from, say, their home city.

When students work with the mileage tables, encourage them to use trial-and-error to find the "itinerary" that is as close to the computed number of miles (above or below) as possible. Remind students that the cities must be *connecting cities* (such as Chicago to New York, New York to London, etc.) You may want to present this as a contest where students compete to come closest to the computed number of miles. For an additional challenge, you may want to require student "itineraries" to include at least, say, four cities.

Distances provided in mileage tables will vary among sources. The source for the mileage tables included with this activity lesson is the publication *infoplease* (http://www.infoplease.com/ipa/A0004594.html and http://www.infoplease.com/ipa/A0759496.html) for U.S. cities and world cities, respectively. Additional mileage tables are available at those two sites.

Students are able to complete this activity lesson without knowledge on how the probabilities are computed. The following mathematical background is provided for teacher edification purposes—and for students who are capable of understanding the probability concepts involved.

Determining lottery probabilities involves finding the total number of combinations of groups of numbers (of specific sizes) that are possible. Since the *order* in which the numbers are drawn does not matter, lottery probabilities are based on *combinations* (unordered groupings) rather than, say, on *permutations* (where order *does* make a difference). The formula involves the use of *factorials*—such as $n!$, read "n factorial." The *factorial* of a positive integer n is the product of all integers from n to 1. So $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

> Formula for finding the number of combinations of n things taken r at a time:
> $${}_nC_r = \frac{n!}{(n-r)! \cdot r!}$$

Illinois Lotto: This lottery uses 52 lottery balls numbered from 1–52. With each \$1 ticket, players choose two combinations of six numbers each. At each drawing, a set of six winning numbers is drawn. To win the jackpot, a player must match all six numbers. The number of combinations of 52 things taken 6 at a time is calculated below:

$${}_{52}C_6 = \frac{52!}{(52-6)! \cdot 6!} = \mathbf{20{,}358{,}520}$$

There are 20,358,520 combinations in all, so with 2 chances of winning on a \$1 ticket, a player has **1 chance in 10,179,260** to win the jackpot.

Powerball: For a \$1 ticket, a player selects numbers from two sets of numbers. The player selects five whole numbers from 1–59, and one whole number, the *Powerball,* from 1–39. To win the jackpot, a player must match all six numbers. The total number of possible combinations is found by multiplying the number of combinations of 59 things taken 5 at a time by the number of combinations of 39 things taken 1 at a time:

$${}_{59}C_5 \times {}_{39}C_1 = \frac{59!}{(59-5)! \cdot 5!} \times \frac{39!}{(39-1)! \cdot 1!} =$$

$$\mathbf{5{,}006{,}386 \times 39 = 195{,}249{,}054}$$

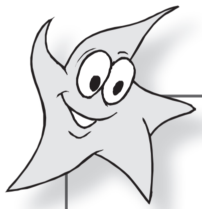
So, on a \$1 ticket, the probability of winning the *Powerball* jackpot is **1 in 195,249,054.**

Mega Millions: For a \$1 ticket, a player selects numbers from two sets of numbers. The player selects five whole numbers from 1–56, and one whole number, the *Mega Ball,* from 1–46. To win the jackpot, a player must match all six numbers. The total number of possible combinations is found by multiplying the number of combinations of 56 things taken 5 at a time by the number of combinations of 46 things taken 1 at a time.

$${}_{56}C_5 \times {}_{46}C_1 = \frac{56!}{(56-5)! \cdot 5!} \times \frac{46!}{(46-1)! \cdot 1!} =$$

$$\mathbf{3{,}819{,}816 \times 46 = 175{,}711{,}536}$$

So, on a \$1 ticket, the probability of winning the *Mega Millions* jackpot is **1 in 175,711,536.** (What's "mega" about this game are the mega-odds against winning!)



Mathematical Humor

Probability of winning the *Powerball* jackpot with 1 ticket:

0.000000005

Probability of winning the *Powerball* jackpot with no tickets:

0.000000000

So, you are about as likely to win *without* a ticket as you are *with a ticket.*

To further illustrate the remote chance of winning the jackpot, have students read the number 0.000000005 (5 *billionths;* $1 \div 195{,}249{,}054 \approx 0.000000005$). So, with 1 ticket, a person has about 5 chances in 1 billion of winning. Thus, if a person were to purchase 10 tickets instead of 1 ticket, that person's chances of winning would improve tenfold to about 0.00000005 (5 chances in 100 million). If a person were to purchase 100 tickets instead of 1 ticket, that person would have about 5 chances in 10 million of winning. Despite these terrible probabilities, it is not uncommon to hear of people who "invest" $100 or more in a given lottery. These people often say, "You have to play to win." But in reality, "You have to play to lose."

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Question: How do you read "5!"?

Answer: **Five!**

Solutions

1a. Determining the number of inches of cups:
3 in. × 195,249,054 = 585,747,162 in.

Converting inches to feet:
585,747,162 in. ÷ 12 ≈ 48,812,264 ft

Converting feet to miles:
48,812,264 ft ÷ 5,280 ≈ 9,245 mi

b. "Itineraries" will vary. One possible solution:
Cleveland to Los Angeles (2,049 mi);
Los Angeles to Mexico City (1,589 mi);
Mexico City to Montreal (2,318 mi);
Montreal to London (3,282 mi);
Total length of cups: 9,238 mi

2a. Determining the number of inches of cups:
3 in. × 175,711,536 = 527,134,608 in.

Converting inches to feet:
527,134,608 in. ÷ 12 = 43,927,884 ft

Converting feet to miles:
43,927,884 ft ÷ 5,280 ≈ 8,320 mi

b. "Itineraries" will vary. One possible solution:
Indianapolis to Chicago (165 mi);
Chicago to Shanghai (7,061 mi);
Shanghai to Manila (1,150 mi).
Total trip: 8,376 mi

c. Answers will vary. To determine the number of round trips, students may divide 8,320 by the distance between the two cities, and then divide by 2. Another option is to multiply the distance between the two cities by 2, and then divide 8,320 by that product.

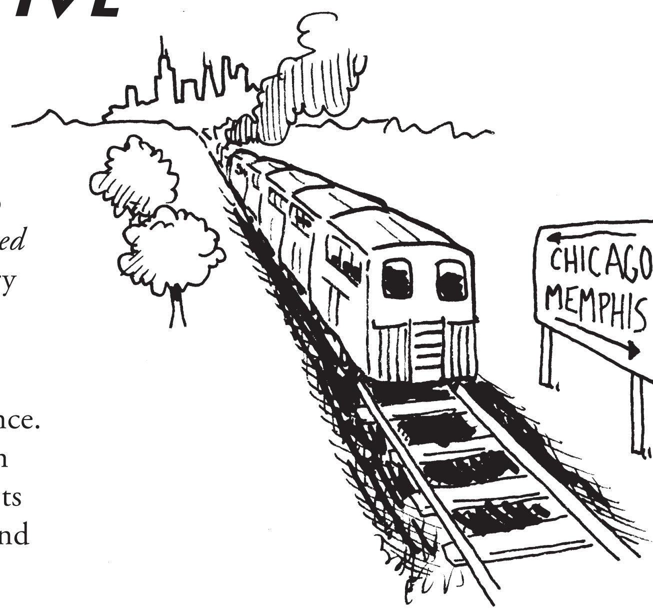
3. Answers will vary. Most students should conclude that it is nearly impossible to win the jackpot in one of these lotteries; hence, it is *not* worth traveling any distance to buy tickets.

Extensions

1. Have students use a globe, map, or other visual to present to the class their route for their cups for each lottery. The presentation should include a discussion of some of the cities, states, bodies of water, and countries that the trip goes through. (A student once commented, "It's going to be tough finding the 'winning cup'—since it is likely to be in the middle of the ocean!")
2. Have students repeat the activity lesson using cups with a metric dimension, say, 5 centimeters wide in diameter. Have students compute the length of the cups in centimeters, and then convert to kilometers (by dividing by 100,000). Have students convert the number of kilometers to miles by multiplying by 0.62.

Name____ Date ____________________

Using Measurement to Put Lottery Probabilities into *Perspective*



Materials: globes or maps (optional)

When someone says, "You're more likely to be struck by lightning than you are to win a lottery," most people are not *shocked* or *jolted.* It's not *striking* news that lottery probabilities are very tiny.

Because lottery probabilities are so tiny, they are outside of our everyday experience. Yet some people spend a lot of money on lotteries. This activity lesson uses concepts from measurement to help us visualize and "put into perspective" probabilities from three different lotteries.

An Example: The *Illinois Lotto*

The *Illinois Lotto* uses 52 lottery balls numbered from 1–52. With each $1 ticket, players choose two combinations of six numbers each. At each drawing, a set of six winning numbers is drawn. To win the jackpot, a player needs to match all six numbers. (Players do not have to match numbers in the order drawn.)

With a $1 ticket, the probability of matching all six numbers is 1 in 10,179,260. To help get a picture of what 1 chance in 10,179,260 might look like, let's imagine how far 10,179,260 objects placed in a line might stretch. Since this model, or representation, involves using a "line," we call the model a *linear representation.*

Linear Representation of the *Illinois Lotto* Jackpot Probability:

Think of placing 10,179,260 paper cups, each 3 inches in diameter, in a straight line.

(a) How many miles long would that be?

(b) Between which two cities might this line of cups extend?

(c) How does this relate to your chances of winning the *Illinois Lotto* jackpot?

Answers to Example on the *Illinois Lotto:*

(a) 10,179,260 cups, each 3 inches in diameter, would extend 3 × 10,179,260, or 30,537,780 inches. Converting this to feet gives us 30,537,780 ÷ 12, or 2,544,815 feet. Dividing this result by 5,280 yields about 482 miles.

(b) This is the approximate distance between Chicago and Memphis!

(c) Suppose someone randomly places a pea under one of those cups—and you are given ONE chance to select the cup with the pea. Your chance of selecting that cup is the same as your chance of winning the *Lotto* jackpot on a $1 play.

Questions

Powerball

Powerball is a lottery played in 42 states, Washington, D.C., and the U.S. Virgin Islands.

For a $1 ticket, a player selects numbers from two sets of numbers. The player selects five whole numbers from 1–59, and one whole number, the *Powerball,* from 1–39. To win the jackpot, a player must match all six numbers. For a $1 ticket, the probability of winning the *Powerball* jackpot is 1 in 195,249,054.

1a. How many miles long would a line of 195,249,054 paper cups, each 3 inches in diameter, extend? ____________

b. Use the mileage tables included with this activity lesson, or another source, to map out a distance to represent the line of cups. Record your "itinerary" with mileage distances—going from one connecting city to the next—in the space below. Try to get as close as possible (above or below) to the number of miles in question 1a. If a globe or map is available, you may want to locate the cities in your "itinerary" to get a feel for the distances involved.

Mega Millions

Mega Millions is a lottery that is played in California, Georgia, Illinois, Maryland, Massachusetts, Michigan, New Jersey, New York, Ohio, Texas, Virginia, and Washington.

For a $1 ticket, a player selects numbers from two sets of numbers. The player selects five whole numbers from 1–56, and one whole number, the Mega Ball, from 1–46. A player needs to match all six numbers to win the jackpot. For a $1 ticket, the probability of winning the *Mega Millions* jackpot is 1 in 175,711,536.

2a. How many miles long would a line of 175,711,536 paper cups, each 3 inches in diameter, extend? ___________

b. Use mileage tables to map out a distance to represent the line of cups. Record your "itinerary" with mileage distances—going from one *connecting* city to the next—in the space below. Try to get as close as possible (above or below) to the number of miles in question 2a.

c. Select a city in the United States other than your own. How many *round trips* between your home city (or a city near you) and the selected city would you need to make in order to cover the number of miles determined in question 2a? Round to the nearest tenth. ___________

3. From time-to-time, the jackpot for *Powerball* or *Mega Millions* grows to be in the hundreds of millions of dollars. In order to participate in an out-of-state lottery, some people drive from one state to another. (For example, a person living in Indiana may drive to Illinois to buy a ticket for *Mega Millions.*) In your opinion, is it worth "going to great lengths" to buy a lottery ticket? Explain.

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Air Distances between U.S. Cities in Statute Miles

Cities	Birmingham	Boston	Buffalo	Chicago	Cleveland	Dallas	Denver
Birmingham, Ala.	—	1,052	776	578	618	581	1,095
Boston, Mass.	1,052	—	400	851	551	1,551	1,769
Buffalo, N.Y.	776	400	—	454	173	1,198	1,370
Chicago, Ill.	578	851	454	—	308	803	920
Cleveland, Ohio	618	551	173	308	—	1,025	1,227
Dallas, Tex.	581	1,551	1,198	803	1,025	—	663
Denver, Colo.	1,095	1,769	1,370	920	1,227	663	—
Detroit, Mich.	641	613	216	238	90	999	1,156
El Paso, Tex.	1,152	2,072	1,692	1,252	1,525	572	557
Houston, Tex.	567	1,605	1,286	940	1,114	225	879
Indianapolis, Ind.	433	807	435	165	263	763	1,000
Kansas City, Mo.	579	1,251	861	414	700	451	558
Los Angeles, Calif.	1,802	2,596	2,198	1,745	2,049	1,240	831
Louisville, Ky.	331	826	483	269	311	726	1,038
Memphis, Tenn.	217	1,137	803	482	630	420	879
Miami, Fla.	665	1,255	1,181	1,188	1,087	1,111	1,726
Minneapolis, Minn.	862	1,123	731	355	630	862	700
New Orleans, La.	312	1,359	1,086	833	924	443	1,082
New York, N.Y.	864	188	292	713	405	1,374	1,631
Omaha, Nebr.	732	1,282	883	432	739	586	488
Philadelphia, Pa.	783	271	279	666	360	1,299	1,579
Phoenix, Ariz.	1,456	2,300	1,906	1,453	1,749	887	586
Pittsburgh, Pa.	608	483	178	410	115	1,070	1,320
St. Louis, Mo.	400	1,038	662	262	492	547	796
Salt Lake City, Utah	1,466	2,099	1,699	1,260	1,568	999	371
San Francisco, Calif.	2,013	2,699	2,300	1,858	2,166	1,483	949
Seattle, Wash.	2,082	2,493	2,117	1,737	2,026	1,681	1,021
Washington, D.C.	661	393	292	597	306	1,185	1,494

Source: From infoplease. Available at http://www.infoplease.com/ipa/A0004594.html. Used with permission. Additional mileage tables for distances between U.S. cities are available at the above Web site.

Air Distances between U.S. Cities in Statute Miles

Cities	Detroit	El Paso	Houston	Indianapolis	Kansas City	Los Angeles	Louisville
Birmingham, Ala.	641	1,152	567	433	579	1,802	331
Boston, Mass.	613	2,072	1,605	807	1,251	2,596	826
Buffalo, N.Y.	216	1,692	1,286	435	861	2,198	483
Chicago, Ill.	238	1,252	940	165	414	1,745	269
Cleveland, Ohio	90	1,525	1,114	263	700	2,049	311
Dallas, Tex.	999	572	225	763	451	1,240	726
Denver, Colo.	1,156	557	879	1,000	558	831	1,038
Detroit, Mich.	—	1,479	1,105	240	645	1,983	316
El Paso, Tex.	1,479	—	676	1,264	839	701	1,254
Houston, Tex.	1,105	676	—	865	644	1,374	803
Indianapolis, Ind.	240	1,264	865	—	453	1,809	107
Kansas City, Mo.	645	839	644	453	—	1,356	480
Los Angeles, Calif.	1,983	701	1,374	1,809	1,356	—	1,829
Louisville, Ky.	316	1,254	803	107	480	1,829	—
Memphis, Tenn.	623	976	484	384	369	1,603	320
Miami, Fla.	1,152	1,643	968	1,024	1,241	2,339	919
Minneapolis, Minn.	543	1,157	1,056	511	413	1,524	605
New Orleans, La.	939	983	318	712	680	1,673	623
New York, N.Y.	482	1,905	1,420	646	1,097	2,451	652
Omaha, Nebr.	669	878	794	525	166	1,315	580
Philadelphia, Pa.	443	1,836	1,341	585	1,038	2,394	582
Phoenix, Ariz.	1,690	346	1,017	1,499	1,049	357	1,508
Pittsburgh, Pa.	205	1,590	1,137	330	781	2,136	344
St. Louis, Mo.	455	1,034	679	231	238	1,589	242
Salt Lake City, Utah	1,492	689	1,200	1,356	925	579	1,402
San Francisco, Calif.	2,091	995	1,645	1,949	1,506	347	1,986
Seattle, Wash.	1,938	1,376	1,891	1,872	1,506	959	1,943
Washington, D.C.	396	1,728	1,220	494	945	2,300	476

Source: From infoplease. Available at http://www.infoplease.com/ipa/A0004594.html. Used with permission. Additional mileage tables for distances between U.S. cities are available at the above Web site.

Air Distances between World Cities in Statute Miles

Cities	Berlin	Buenos Aires	Cairo	Calcutta	Cape Town	Caracas	Chicago
Berlin	—	7,402	1,795	4,368	5,981	5,247	4,405
Buenos Aires	7,402	—	7,345	10,265	4,269	3,168	5,598
Cairo	1,795	7,345	—	3,539	4,500	6,338	6,129
Calcutta	4,368	10,265	3,539	—	6,024	9,605	7,980
Cape Town	5,981	4,269	4,500	6,024	—	6,365	8,494
Caracas	5,247	3,168	6,338	9,605	6,365	—	2,501
Chicago	4,405	5,598	6,129	7,980	8,494	2,501	—
Hong Kong	5,440	11,472	5,061	1,648	7,375	10,167	7,793
Honolulu	7,309	7,561	8,838	7,047	11,534	6,013	4,250
Istanbul	1,078	7,611	768	3,638	5,154	6,048	5,477
Lisbon	1,436	5,956	2,363	5,638	5,325	4,041	3,990
London	579	6,916	2,181	4,947	6,012	4,660	3,950
Los Angeles	5,724	6,170	7,520	8,090	9,992	3,632	1,745
Manila	6,132	11,051	5,704	2,203	7,486	10,620	8,143
Mexico City	6,047	4,592	7,688	9,492	8,517	2,232	1,691
Montreal	3,729	5,615	5,414	7,607	7,931	2,449	744
Moscow	1,004	8,376	1,803	3,321	6,300	6,173	4,974
New York	3,965	5,297	5,602	7,918	7,764	2,132	713
Paris	545	6,870	1,995	4,883	5,807	4,736	4,134
Rio de Janeiro	6,220	1,200	6,146	9,377	3,773	2,810	5,296
Rome	734	6,929	1,320	4,482	5,249	5,196	4,808
San Francisco	5,661	6,467	7,364	7,814	10,247	3,904	1,858
Shanghai	5,218	12,201	5,183	2,117	8,061	9,501	7,061
Stockholm	504	7,808	2,111	4,195	6,444	5,420	4,278
Sydney	10,006	7,330	8,952	5,685	6,843	9,513	9,272
Tokyo	5,540	11,408	5,935	3,194	9,156	8,799	6,299
Warsaw	320	7,662	1,630	4,048	5,958	5,517	4,667
Washington, D.C.	4,169	5,218	5,800	8,084	7,901	2,059	597

Source: From infoplease. Available at http://www.infoplease.com/ipa/A0759496.html. Used with permission. Additional mileage tables for distances between world cities are available at the above Web site.

Air Distances between World Cities in Statute Miles

Cities	Hong Kong	Honolulu	Istanbul	Lisbon	London	Los Angeles	Manila
Berlin	5,440	7,309	1,078	1,436	579	5,724	6,132
Buenos Aires	11,472	7,561	7,611	5,956	6,916	6,170	11,051
Cairo	5,061	8,838	768	2,363	2,181	7,520	5,704
Calcutta	1,648	7,047	3,638	5,638	4,947	8,090	2,203
Cape Town	7,375	11,534	5,154	5,325	6,012	9,992	7,486
Caracas	10,167	6,013	6,048	4,041	4,660	3,632	10,620
Chicago	7,793	4,250	5,477	3,990	3,950	1,745	8,143
Hong Kong	—	5,549	4,984	6,853	5,982	7,195	693
Honolulu	5,549	—	8,109	7,820	7,228	2,574	5,299
Istanbul	4,984	8,109	—	2,012	1,552	6,783	5,664
Lisbon	6,853	7,820	2,012	—	985	5,621	7,546
London	5,982	7,228	1,552	985	—	5,382	6,672
Los Angeles	7,195	2,574	6,783	5,621	5,382	—	7,261
Manila	693	5,299	5,664	7,546	6,672	7,261	—
Mexico City	8,782	3,779	7,110	5,390	5,550	1,589	8,835
Montreal	7,729	4,910	4,789	3,246	3,282	2,427	8,186
Moscow	4,439	7,037	1,091	2,427	1,555	6,003	5,131
New York	8,054	4,964	4,975	3,364	3,458	2,451	8,498
Paris	5,985	7,438	1,400	904	213	5,588	6,677
Rio de Janeiro	11,021	8,285	6,389	4,796	5,766	6,331	11,259
Rome	5,768	8,022	843	1,161	887	6,732	6,457
San Francisco	6,897	2,393	6,703	5,666	5,357	347	6,967
Shanghai	764	4,941	4,962	6,654	5,715	6,438	1,150
Stockholm	5,113	6,862	1,348	1,856	890	5,454	5,797
Sydney	4,584	4,943	9,294	11,302	10,564	7,530	3,944
Tokyo	1,794	3,853	5,560	6,915	5,940	5,433	1,866
Warsaw	5,144	7,355	863	1,715	899	5,922	5,837
Washington, D.C.	8,147	4,519	5,215	3,562	3,663	2,300	8,562

Source: From infoplease. Available at http://www.infoplease.com/ipa/A0759496.html. Used with permission. Additional mileage tables for distances between world cities are available at the above Web site.

Discovering a Formula for the Area of a Circle

Teacher's Notes

NCTM Standards:	Geometry; Algebra
CCS Standards:	Geometry; The Number System; Look for and express regularity in repeated reasoning.
Mathematical Topics:	Estimate area by counting square units; find the area of a square, of a triangle, and of a circle; explore pi; find a pattern; use formulas
Grouping of Students:	Work in pairs or individually

Background

Without geometry, life would be pointless.

—Source unknown

A circle is defined as a set of points in a plane that are all the same distance from a given point called the *center.* In this activity lesson, students "discover" a formula for finding the area of a circle by counting square units inside a circle and comparing that estimate to the area of a circumscribed square and of an inscribed square. Students examine the data and conclude that the area of the circle is somewhere between the areas of those two squares. Students discover that the formula involves squaring the radius and multiplying by a factor that is slightly greater than 3—which is the key result of the activity lesson. The activity lesson concludes with a review of the number π—and its use in the formula for the area of a circle (as the constant number that is slightly greater than 3).

After students estimate the area of each circle by counting square units, have them compare their estimates with those of other students. It's important that students establish a reasonable estimate for the area of the circle before moving on to comparing that area to the area of the squares.

The activity lesson assumes that students have been exposed to finding the circumference of a circle and to the number π, although this is not a requirement. If students have not previously explored the circumference of a circle with actual objects, you may want to do the following activity with them prior to this activity lesson:

Have students use a tape measure to find the distance around various-sized circles (the circumference) that are formed by cans and other objects. Have them also measure the distance across each circle (the diameter). Then have students find the following ratio for each circle, rounded to the nearest hundredth:

$$\frac{\text{distance around}}{\text{distance across}}$$

Ask students the following question:

> *As you consider larger and larger circles, does the ratio become larger, stay about the same, or become smaller?*

Students should conclude that it stays about the same—at about 3.14. Advise students that we call this number pi, denoted π. It is the ratio of the circumference of a circle to its diameter—and it is always the same number, no matter what size circle you consider.

Unlike a number such as 5.78 that has a finite number of nonzero digits to the right of the decimal point, π has infinitely many digits to the right of the decimal point—and the digits do not repeat. We write $\pi = 3.14159265\ldots$, and we say that π is an *irrational number.* (A *rational number* is a number that can be written as a simple fraction $\frac{a}{b}$, where $b \neq 0$. Rational numbers include both decimals that terminate and decimals that repeat.)

Pi is a number that has been known for a long, long time. The Old Testament alludes to a value of 3 for pi in 1 Kings 7:23 (probably suggesting just an estimate). The ancient Egyptians and the Babylonians knew about the existence of the constant ratio pi. They figured out that it was a number that is slightly greater than 3. The Babylonians had an approximation of $3\frac{1}{8}$ (3.125); the Egyptians used $4 \bullet (\frac{8}{9})^2$ (about 3.160494). The sixteenth letter of the Greek alphabet, π, was first used for the value 3.1415 by Welsh mathematician William Jones in 1706. The Greek letter pi was chosen as the letter to represent this ratio because the letter π in Greek stands for "perimeter." (The perimeter of a circle is the same as its circumference.)

There are a number of Web sites that provide information on pi and suggestions for celebrating Pi Day (March 14). These include:

- The Joy of Pi (Pi Pages on the Internet): http://www.joyofpi.com/pilinks.html
- The Math Forum: http://mathforum.org/dr.math/faq/faq.pi.html

Mathematical Humor

Question: What do you get if you divide the circumference of a jack-o-lantern by its diameter?
Answer: pumpkin pie

Question: What do you get if you divide an igloo's circumference by its diameter?
Answer: Eskimo pi

Question: What do you get if you divide the circumference of the sun by its diameter?
Answer: pi in the sky

One could say that we are really *pi-ling* on the bad jokes here.

Solutions

1. Answers will vary. Many students may conclude that the area is somewhere between 78 and 79 square units.
2. 100 square units
3. Each triangle has an area of $\frac{1}{2} \bullet 10 \bullet 5$, or 25 square units. So the area of the tilted square is 2×25, or 50, square units.
4. **3** × 5 × 5 = **75** square units
5. Most students should conclude, "Yes."

6a. Answers will vary. Many students should conclude that the area is somewhere between 50 and 51 square units.

a. 48 square units

b. Most students should conclude, "Yes."

7. Most students should conclude that in each case, the Calculated Area is less than the Estimated Area. For the circle with a radius of 5, the Calculated Area is 75 square units (versus an Estimated Area of about 78 to 79 square units); for the circle with a radius of 4, the Calculated Area is 48 square units (versus an Estimated area of about 50 to 51 square units). So in the formula, we should use a number that is slightly greater than 3.

8a. 78.5 square units

b. 50.24 square units

9. Students should conclude that $A = 3.14 \times r \times r$ provides a reasonable formula for finding the area of a circle. The computed areas with that formula (78.5 and 50.24 square units, respectively) are very close to the Estimated Areas based on counting the actual square units (78 to 79 and 50 to 51 square units, respectively).

Extension

Linking the Digits of Pi: Have students work together to make a long paper chain, joining links representing the first 200 or more digits of pi. Use ten different colors, one for each of the digits 0–9.

For example, you could have 0 = white, 1 = purple, 2 = green, 3 = red, 4 = yellow, and so on. The chain would begin with: a red link (for 3), a purple link (for 1), a yellow link (for 4), and so on.

The first 50,000 digits of pi are posted at http://www.ballandclaw.com/upi/pi.50000.html. Good luck.

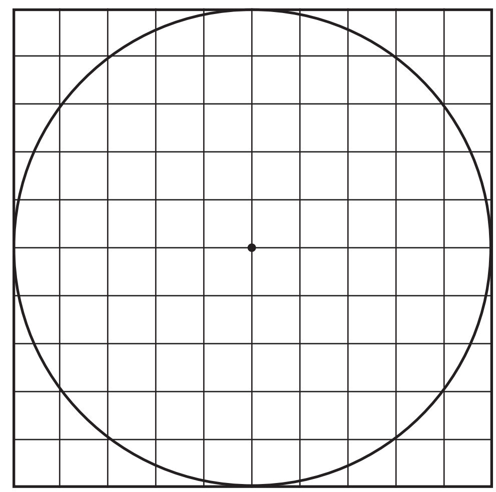
You could also have students make a frequency table and a bar graph—or even a pi PIE graph—to display the distribution of pi's digits that are represented in the chain. Students could also compute the *mean, median,* and the *mode* of the digits.

Name____ Date ____________________

Discovering a Formula for the Area of a Circle

1. Estimate the area of the circle at right by counting squares and partial squares. The circle has a radius of 5 units.

   Estimated Area of Circle = ______________



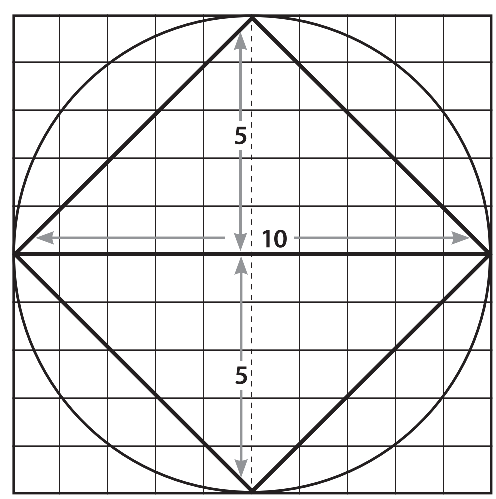
2. Find the area of the entire 10-by-10 grid at right.

   Area of Entire Grid = _________________

3. Find the area of the tilted square that is inside the circle at right. To do so, find the area of each of the two large triangles, and then add the areas.

   Recall that to find the area of a triangle, multiply the base by the height, and divide by 2. (The formula is $A = \frac{1}{2} b \bullet h$.) Each large triangle has a base of 10 units and a height of 5 units.

   Area of Tilted Square Inside Circle = _________________



Recall the story "Goldilocks and the Three Bears." An important lesson in the story is that if something is "too big" or "too small," then "just right" is somewhere in between. We will use this line of thinking to help us discover a *formula* to *calculate* the area of a circle.

Consider the following data:

Too Big	Area of Entire Grid	100 square units
Just Right	Area of Circle	
Too Small	Area of Tilted Square Inside Circle	50 square units

4. To help us discover a formula for the area of a circle, look for a pattern in the data. Recall that a *radius* of the circle is 5 units. Try to rewrite the numbers in the above table by using the number 5 as a factor (because it would be nice to have the radius be part of the formula).

   Notice below that we can rewrite 100 as $4 \times 5 \times 5$, and we can rewrite 50 as $2 \times 5 \times 5$. Look at the table below. What factor do you think we should use to complete the "Just Right" row? Record the factor in the table, and write the product.

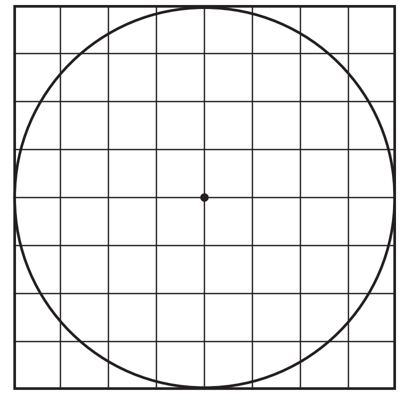
Too Big	Area of Entire Grid	4 x 5 x 5 = 100 square units
Just Right	Area of Circle	____ x 5 x 5 = ____ square units
Too Small	Area of Tilted Square Inside Circle	2 x 5 x 5 = 50 square units

5. Compare the area you calculated in the "Just Right" row above to your Estimated Area of the Circle in question #1. Is your calculated area above close to the Estimated Area? ____________

So, it appears that $3 \times 5 \times 5$ provides a good estimate for the area of the circle. Since 5 is a radius of the circle, we could write a formula (general rule) for the area of a circle as follows: Area = $3 \times$ radius $\times$ radius. This can be shortened to $A = 3 \times r \times r$.

Let's test our formula with another circle.

6. The circle at the right has a radius of 4 units. *Estimate* the area of the circle by counting squares and partial squares. Then use your formula to *calculate* the area.



a. Estimated Area: ___________

b. Calculated Area: ___________

c. Is your Calculated Area close to the Estimated Area? ___________

7. In each of questions 4 and 6, was your Calculated Area *greater than* or was it *less than* the Estimated Area? How might you adjust the formula so that it produces a result that is closer to the Estimated Area?

____

____

____

There is a number that is slightly greater than 3 that is used in the formula for the area of a circle. The number is **3.1415916** . . . This number, named $\boldsymbol{\pi}$, is a decimal that does not end and its decimal expansion does not repeat. You may recall that π is the ratio of the circumference to the diameter of any circle. For convenience, we generally use **3.14** for this number.

8. Use the formula $A = 3.14 \times r \times r$ to find the area of each of these circles:

a. circle with a radius of 5 units ___________

b. circle with a radius of 4 units ___________

c. Compare the areas in questions 8a and 8b to your Estimated Areas for those circles. Based on your results, does $A = 3.14 \times r \times r$ provide a good formula for finding the area of a circle? Explain.

____

____

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Is Pythagoras in the *Area?*
Discovering a Famous Relationship
Teacher's Notes

- **NCTM Standards:** Geometry; Algebra; Measurement
- **CCS Standards:** Geometry; Expressions and Equations;Look for and express regularity in repeated reasoning.
- **Mathematical Topics:** Find the area of triangles and squares; look for a pattern; use the Pythagorean Theorem; evaluate expressions; measure to the nearest millimeter
- **Grouping of Students:** Work in small groups or individually

Background

The sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the remaining side.

—Erroneous statement by the scarecrow upon receiving his "brain" in *The Wizard of Oz* (He should have said, "The square root of the sum of the squares of the two legs of a right triangle is equal to the length of the hypotenuse.")

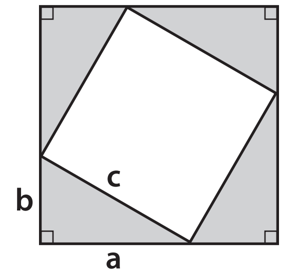
In this activity lesson, students discover a pattern (make a conjecture) on the basis of examining area relationships of three different sets of triangles that are placed around a square. Based on this pattern (the Pythagorean Theorem), students determine a method for finding the hypotenuse when given the two legs of a right triangle. Students test their method by drawing their own right triangle, measuring the legs, and then using their method to determine the hypotenuse. Students then measure the hypotenuse to see if their method worked for the triangle that they drew. Students are then told that their pattern is known as the Pythagorean Theorem.

Although these instances should be convincing to students, the conclusions drawn do not constitute a *proof* that the Pythagorean Theorem works for *any* right triangle. There are many different proofs of the Pythagorean Theorem. Over 90 of them are provided at the following Web site: http://www.cut-the-knot.org/pythagoras/index.shtml.

You may want to present students with the above quote by the scarecrow from The *Wizard of Oz.* Ask students to write correct dialogue for the scarecrow.

The approach used in this activity lesson was inspired by the following popular proof of the Pythagorean theorem.

Proof



a. Each side of the large square is $a + b$ units long.
So the area of the large square is $(a + b)^2 = a^2 + 2ab + b^2$.

b. The area of each of the right triangles is $\frac{ab}{2}$.
So, the area of all four triangles is $4 \bullet \frac{ab}{2}$, or $2ab$.

c. The area of the tilted square is c^2.

d. We can show the area of the large square as the sum of its parts: $2ab + c^2$.

e. We now have two expressions (steps a and d) for the area of the large square.
Setting the two expressions equal to each and simplifying yields:

$a^2 + 2ab + b^2 = 2ab + c^2$

$a^2 + b^2 = c^2$ Subtract $2ab$ from each side.

It is important to note that the statement of the formula $a^2 + b^2 = c^2$ was known long before the days of Pythagoras (who lived approximately 560 B.C.–480 B.C.). In particular, the relationship was discovered on a Babylonian tablet circa 1900–1600 B.C. It is uncertain as to whether or not Pythagoras or one of his colleagues was the first to discover a proof of this relationship.

An electronic activity that provides a model that can be manipulated to show that $a^2 + b^2 = c^2$ is true for any right triangle is available at http://www.cc.k12.nf.ca/sketch/discover20.htm.

Mathematical Humor

Question: What do you call a video of a person placing paper triangles around a square?

Answer: Pa-per view.

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

Question: Why didn't Pythagoras apply his theorem to equilateral triangles?

Answer: Because those triangles are not *right*.

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

The best angle from which to approach a geometry problem is the TRY-angle.

Solutions

1a. 6 square units

b. 24 square units

2a. 7 units (3 units + 4 units)

b. 49 square units

3. 25 square units (49 square units – 24 square units)

4. 5 units

5. The table below has been completed for questions 5, 7, and 8.

6a. 10 square units

b. 40 square units

c. 9 units (4 units + 5 units)

d. 81 square units

e. 41 square units (81 square units – 40 square units)

f. 6.4 units

7. See table below.

8. See table below.

9. The sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse. So, $a^2 + b^2 = c^2$.

10. To find the hypotenuse, compute the sum of the squares of the two legs. Then take the square root. The result is the hypotenuse.

11a.–d. Answers will vary. Students draw a right triangle on grid paper and then measure the lengths of the two legs to the nearest millimeter. Check students' measurements for *a* and *b*. Students should apply their method by computing $a^2 + b^2$ and then taking the square root of that sum (to the nearest millimeter) to determine *c*. Students then measure *c* to the nearest millimeter to verify that their method worked (for this instance). The computed and measured values for *c* need not be identical to verify the method. Since all measurements are approximate, students should be satisfied if their results for *c* are close.

	Length of leg *a*	**Length of leg *b***	**Length of side *c***	**Compute $a^2 + b^2$**	(Area of tilted square in square units) c^2
Data for Square I	3 units	4 units	5 units	25	25
Data for Square II	4 units	5 units	6.4 units	41	41
Data for Square III	6 units	8 units	10 units	100	100

Extension

Pythagorean triples that are not used in this activity lesson include 5, 12, 13; 7, 24, 25; 9, 40, 41; 16, 63, 65; and 20, 21, 29. There are infinitely many Pythagorean triples.

The following proof verifies that the formulas used in the Extension (appearing in the activity lesson) always yield a Pythagorean triple.

Let n and m be any integers, where $n > m$. Define the integers as follows:

$a = n^2 - m^2$

$b = 2nm$

$c = n^2 + m^2$

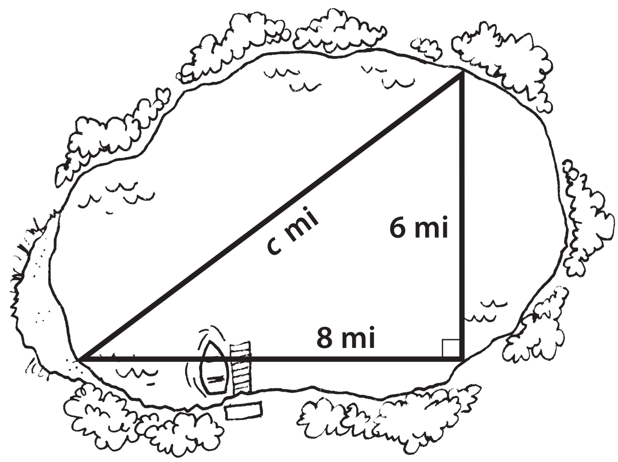
Proof

Find the value of $a^2 + b^2$ in terms of n and m. Check to see that that value is equal to c^2.

$$
\begin{aligned}
\overbrace{(n^2 - m^2)^2}^{a^2} + \overbrace{(2nm)^2}^{b^2} &= n^4 - 2n^2m^2 + m^4 + 4n^2m^2 \\
&= n^4 + 2n^2m^2 + m^4 \\
&= \underbrace{(n^2 + m^2)^2}_{c^2}
\end{aligned}
$$

Name____ Date ___________________

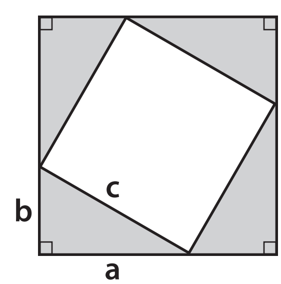
IS PYTHAGORAS IN THE *AREA*?



Materials: scissors; metric ruler

In a right triangle, the two sides that meet to form a right angle are called the *legs.* The side opposite the right angle is called the *hypotenuse.* Usually the variables *a* and *b* are used to represent the legs, and *c* is used to represent the hypotenuse. Often in the real world, we know the lengths of the two legs, but we need to find the length of the hypotenuse (such as finding the distance across a lake, as illustrated at the right).

In this activity lesson, you will . . .



- place four right triangles around a square, as shown at the right, to form a large square,
- find the areas of the right triangles and of the two squares (the large one and the tilted one),
- repeat the procedure for two more sets of data, and
- use the data to find a pattern that will help you discover a formula for finding the length of a hypotenuse when given the values of the two legs of a right triangle.

Cut out the four right triangles with legs 3 units and 4 units from page 123. Then put them around Square I on page 124 as shown above.

1a. What is the area of each of the triangles? ___________

Recall that the formula for the area of a triangle is $\frac{1}{2}$ $b \cdot h$.

b. What is the combined area of all four triangles? ___________

2a. What is the length of each side of the large square?

(The large square is composed of the tilted square and four triangles.) ___________

b. What is the area of the large square? ___________

3. Find the area of the tilted square (Square I).
Hint: Subtract your answer for question 1b from your answer for question 2b. ____________

4. Find the length of side c. (This is a side of Square I and also the hypotenuse of any of the right triangles.)
Hint: Since the formula for the area of a square is $A = s^2$, we can find the length of a side by taking the square root of the square's area. ____________

5. Complete the first row of the table below for Square I.

	Length of leg *a*	Length of leg *b*	Length of side *c*	Compute $a^2 + b^2$	(Area of tilted square in square units) c^2
Data for Square I					
Data for Square II					
Data for Square III					

6. Now cut out the four right triangles with legs 4 units and 5 units from page 123. Place them around Square II on page 124.
   a. What is the area of each of the triangles? ____________
   b. What is the combined area of all four triangles? ____________
   c. What is the length of each side of the large square? ____________
   d. What is the area of the large square? ____________
   e. Find the area of the tilted square (Square II). ____________
   f. Find the length of side *c* to the nearest tenth of a unit. ____________
7. Complete the second row of the table above for Square II.
8. Now cut out the triangle with legs 6 units and 8 units from page 123. Use it to trace four triangles around Square III on page 124 to help you complete the third row of the table.

9. Put on your detective hat and search for a pattern in the table. In each row, what do you notice about the values in the final two columns?

____

____

____

____

10. Use the pattern you discovered in question 9 to develop a method for finding the hypotenuse of a right triangle when given the lengths of the two legs.

____

____

____

11a. Let's test your method. Use a metric ruler to draw a right triangle on the grid paper located at the bottom right-hand corner on page 123. Measure the two legs to the nearest millimeter (tenth of a centimeter). *Do not measure the hypotenuse.*

a = ________ ; b = ________

b. Use your method from question 10 to determine c to the nearest millimeter (tenth of a centimeter). c = ________

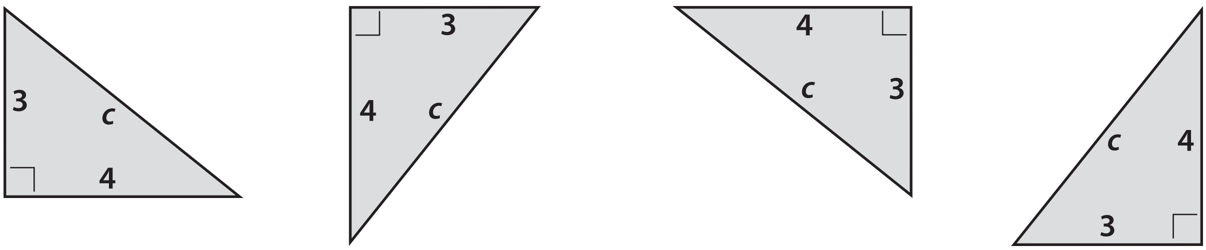
c. Now use your metric ruler to measure c to the nearest millimeter. c = ________

d. How do your computed results and measured results for side c in questions 11b. and 11c. compare?

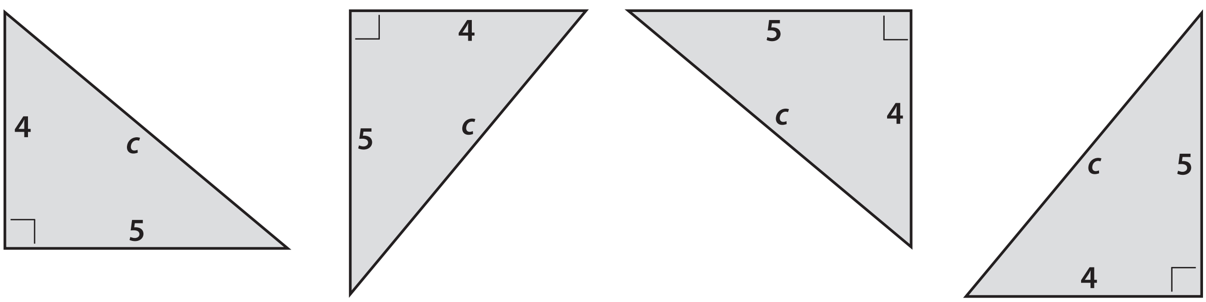
____

The pattern you discovered in question 9 is known as the *Pythagorean Theorem.* It was named after the Greek philosopher and mathematician Pythagoras (who lived approximately 560 B.C.–480 B.C.) because he wrote extensively on the subject.

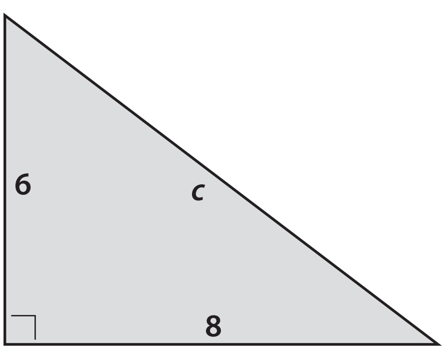
Place these triangles around Square I.

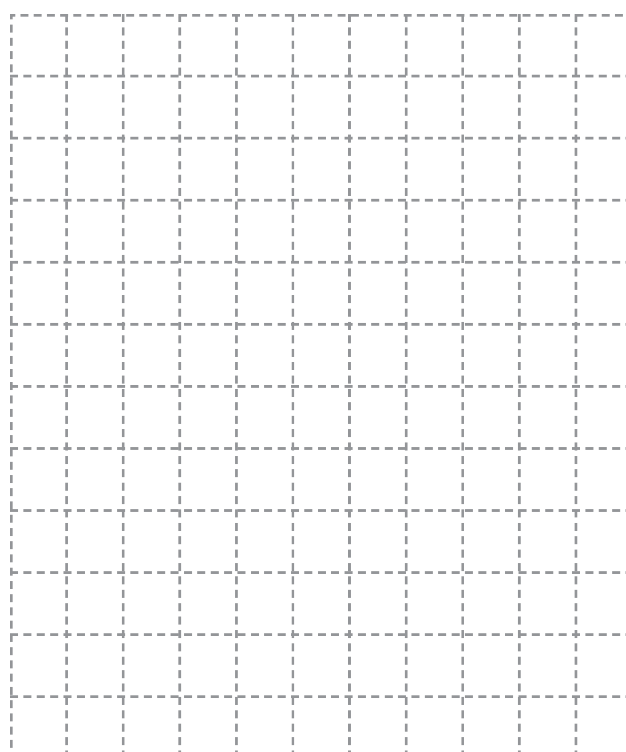


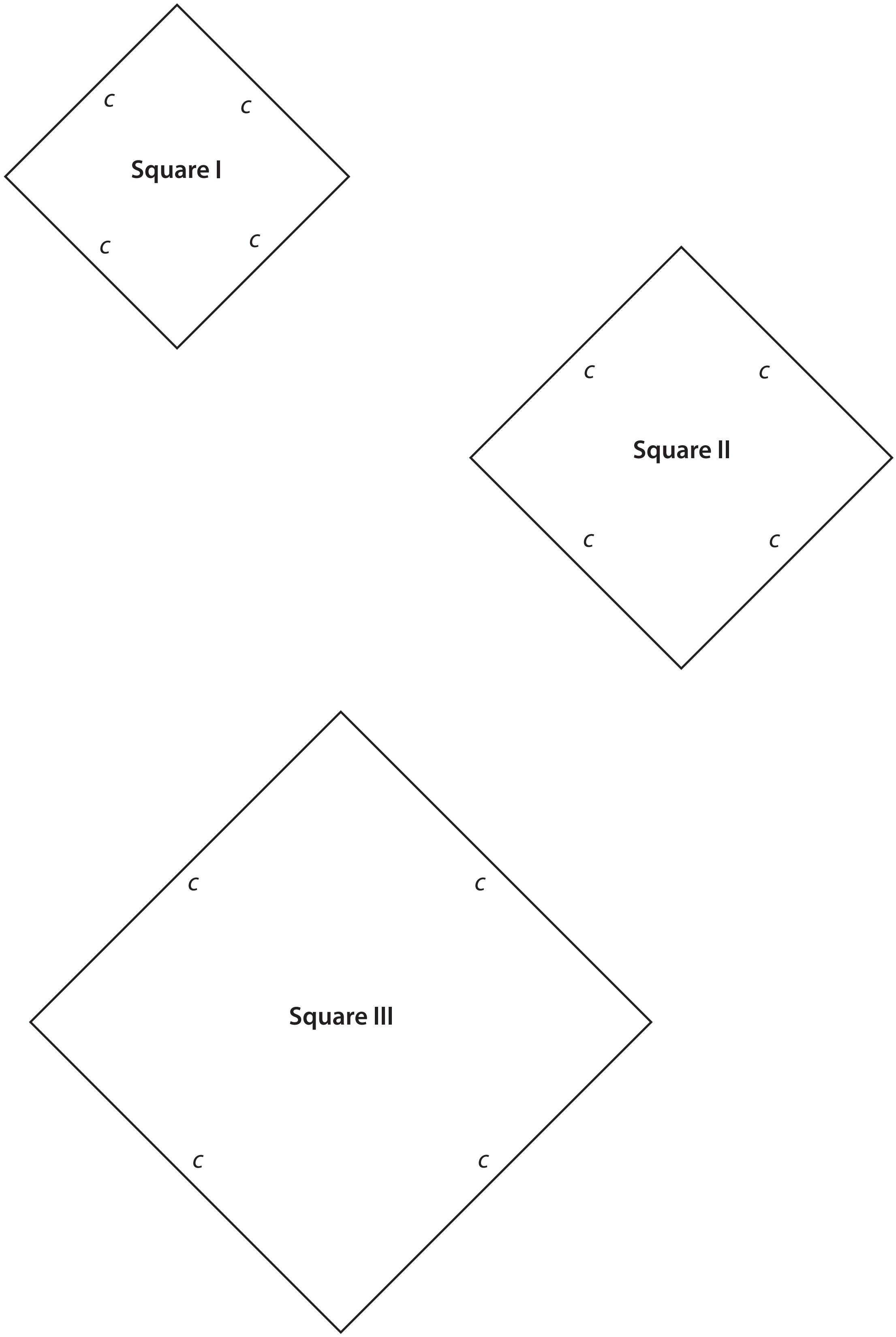
Place these triangles around Square II.



Trace this triangle around Square III.






Square I
c
c
c
c
Square II
c
c
c
c
Square III
c
c
c
c

Extension

When all three sides of a right triangle are integers, those integers form what is known as a *Pythagorean triple.* There are infinitely many Pythagorean triples. Use the following formula to generate them:

Let n and m be any integers, where $n > m$. Define the integers as follows:

$a = n^2 - m^2$

$b = 2nm$

$c = n^2 + m^2$

The three numbers a, b, and c always form a Pythagorean triple.

Use the above formulas to find four sets of Pythagorean triples (that have not already been used in this activity lesson). Then test them (with the rule you proposed in Question 9) to verify that they are correct.

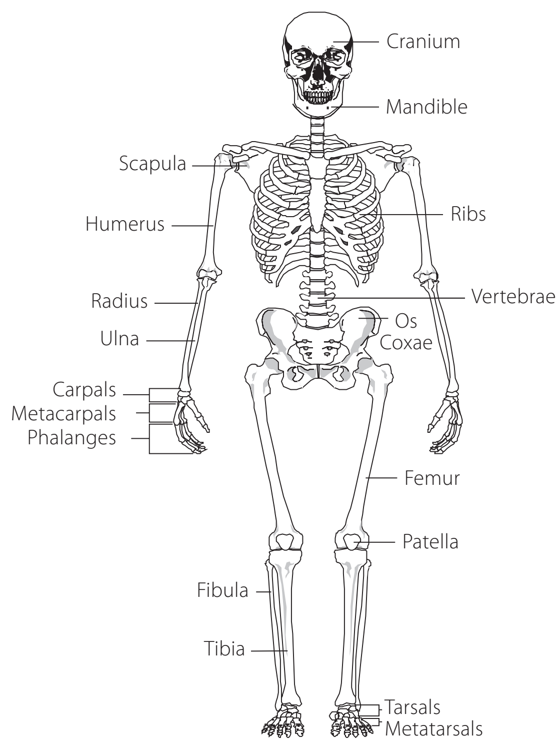
Pythagorean triple: ___________ ___________ ___________

Pythagorean triple: ___________ ___________ ___________

Pythagorean triple: ___________ ___________ ___________

Pythagorean triple: ___________ ___________ ___________

4 Focus on Algebra



All students should learn algebra.

—Principles and Standards for School Mathematics (NCTM, 2000)

When Will Scruffy Be as Old as Joey?*

Investigating a Problem from Three Perspectives

Teacher's Notes

- **NCTM Standards:** Number and Operations, Data Analysis, Algebra
- **CCS Standards:** Expressions and Equations; Functions; Reason abstractly and quantitatively.
- **Mathematical Topics:** Investigate three ways to solve a problem: make a table and analyze using trial-and-error, make a dual graph, set up and solve an equation
- **Grouping of Students:** Work in small groups, in pairs, or individually

Background

Students should solve problems in which they use tables, graphs, words, and symbolic expressions to represent and examine functions and patterns of change.

—Principles and Standards for School Mathematics (NCTM, 2000)

In this activity lesson, students solve a real-world problem using three different solution methods and representations. The different solution methods illustrate an interconnectivity among several strands of mathematics while at the same time suggesting a clear transition from concrete aspects of arithmetic to more abstract aspects of algebra.

Making connections between arithmetic and algebra should begin at the earliest grades. One key area where such connections should be made is relating work with completing number facts (see below) with the solving of simple equations.

$$c + 3 = 11$$

Many young students think the = sign separates "the problem" from "the solution." They see the = sign as a signal to perform an operation—rather than as a symbol of *equality* and *balance.* As such, some of them conclude that the solution to the above sentence is $c = 14$. Such erroneous thinking is especially common among students who are only exposed to number sentences where the unknown is alone on one side of the equation, as in $8 + 7 = c$. Unfortunately, many children do not change this thinking as they get older. On the other hand, students who have had early experience with *trial-and-error substitution* as a way to solve number sentences understand the balance that is needed between the two sides of a sentence. Later, when these students develop skills in using *inverse operations* to solve equations, they retain this substitution method as a way to check their work.

*This activity lesson is dedicated to the memory of Scruffy, the author's dog who passed away in 2007.

The context of this activity lesson involves the aging of dogs—a topic of great interest among students, especially among those who have pet dogs. Although it is recognized that the aging of dogs is not a constant 7 years per human year, this relationship has been accepted over time and well serves the purposes of this problem.

You may want students to use a graphing calculator as they work through each solution method.

Students will first solve the problem by making and analyzing a table, adding 7 years to Scruffy's age for each year of the Joey's age. Since Scruffy was born when Joey was 21, the table reveals that both have the same age somewhere between Joey's ages of 24 and 25, and Scruffy's ages of 21 and 28. This occurs somewhere between Years 3 and 4 (since the birth of Scruffy). Trial-and-error reasoning reveals that the two will have the same age precisely at the midpoint between Years 3 and 4, or 3.5 years. This occurs when both are 24.5 years old (the midpoints of 24 and 25, and of 21 and 28, respectively).

The second solution method involves students making a dual line graph of the two sets of table values. The two lines, of course, intersect where each is 24.5 years old. It is important that students label all elements of the graph—especially each of the two lines.

The third solution method involves setting up and solving an equation. Key is student understanding that if Joey will be $21 + x$ years old when the two have the same age, Scruffy will be $7x$ years old at that time.

You may want to pose additional questions related to the problem, such as those that follow, and have students choose a solution method. You may also want students to discuss the strengths and limitations for using the various solution methods:

1. In human years, how old will Joey and Scruffy be in 5 years? (Joey 26 years old; Scruffy 35 years old) In 10 years? (Joey 31 years old; Scruffy 70 years old)
2. How old will Joey be when Scruffy's age in human years is 39? (about 26.5 years old) When Scruffy's age in human years is 100? (about 35.3 years old)
3. In how many years will Scruffy's age in human years be twice that of Joey's? How old will each be? (Let x be the number of years it will take. Solving the equation:

   $$2(21 + x) = 7x \text{ yields } x = 8.4$$

   So Joey will be 21 + 8.4, or 29.4 years old; Scruffy will be 7 × 8.4, or 58.8 years old.)

Mathematical Humor

Three dogs are ordering dinner off a menu at a restaurant for dogs.

Dog A: I'd like to have five dog biscuits, please.
Dog B: Please serve me the New York sirloin steak.
Dog C: I'll order the math homework.

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

Student A: My dog knows how to multiply by 0. When I ask, "What is 5 × 0?", she says nothing.

Student B: Well my dog knows how to estimate. When I ask him if we have about enough money to buy something, he gives me a *ruff* answer.

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

Comedian George Burns (1896–1996), commenting on longevity: "If you live to be 100, you'll probably live forever—because very, very few people die past 100."

Comedian Jack Benny (1894–1974), commenting on his "eternal age" of 39: "When I grew up in Waukegan, Illinois, I had a younger sister Florence. Now I have an older sister Florence."

Solutions

I. Make a Table

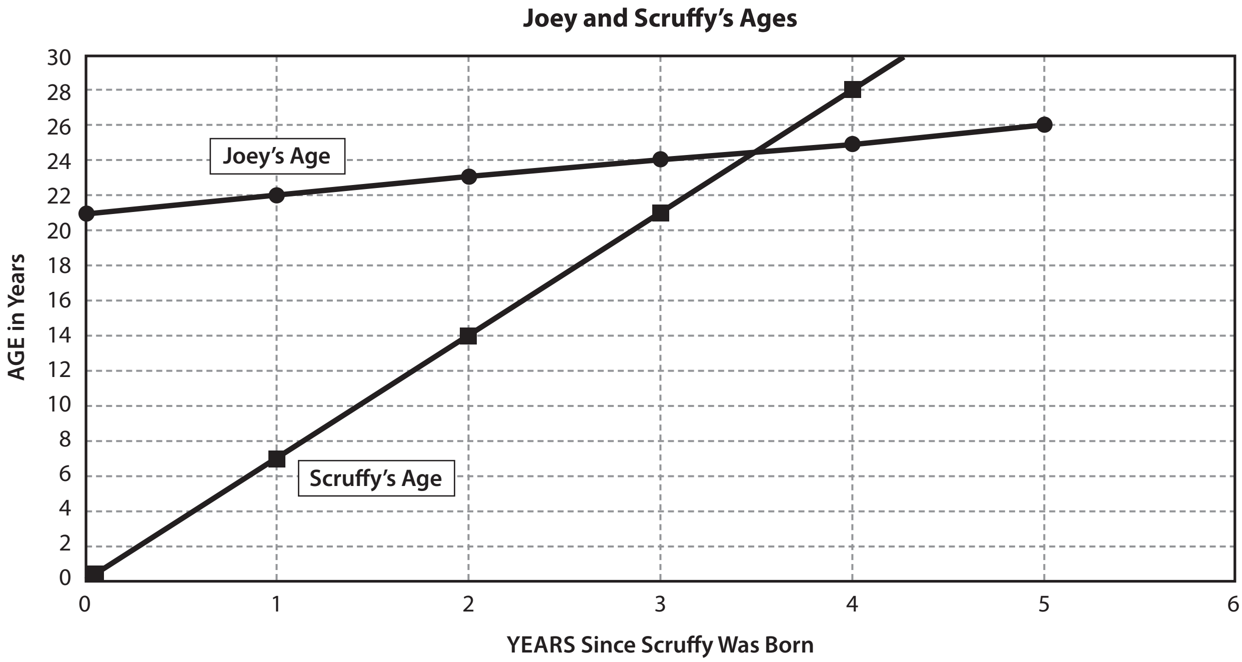
Determining When Scruffy Will Be as Old as Joey

Year	Joey's Age	Scruffy's Age
0	21	0
1	22	7
2	23	14
3	24	21
→ **3.5**	**24.5**	**24.5**
4	25	28

a. Scruffy will be as old as Joey in 3.5 years.

b. Each will be 24.5 years old.

II. Make a Graph



a. Scruffy will be as old as Joey in 3.5 years.   b. Each will be 24.5 years old.

III. Write and Solve an Equation

Let x = the number of years it will take for Scruffy to be as old as Joey.

$21 + x = 7x$

$21 = 6x$

$x = 3.5$

a. Scruffy will be as old as Joey in 3.5 years.

b. Each will be 24.5 years old.

Extension

In the original problem Scruffy is born when Joey is a whole number of years old. In the Extension, Ben's age is a few days beyond a whole number of years when Scruffy is born. Hence, calculations need to zoom in at the day level. The sample solution provided below assumes a 365-day year.

Ben was 18 years old and 12 days when Scruffy was born; hence Ben was $18\frac{12}{365}$, or about 18.03 years old when Scruffy was born. Let x = the number of years for Scruffy to be as old as Ben.

$18.03 + x = 7x$

$6{,}582 = 6x$

$1{,}097 = x$

$3.005 \approx x$

So, 3.005 years after Scruffy's birthday, Scruffy and Ben will have the same age. Since 0.005 year translates to about 2 days ($0.005 \times 365 = 1.825$), Scruffy and Ben will have the same age 3 years and 2 days after Scruffy's birthday. This occurs on February 14, 2009.

Name____ Date ___________________

When Will Scruffy Be as Old As Joey?

Many dog owners enjoy a special relationship with their four-legged friends. But because dogs age about 7 times as fast as humans (or so it is assumed), this relationship often ends all too soon. At some point in time, the dog will become "older" than its owner. We're going to explore a problem that many dog owners have often pondered.

The Problem:

When Scruffy was born, Joey was 21 years old. For every year Joey ages, Scruffy ages 7 years.

a. In how many years will Scruffy be as exactly as old as Joey?
b. How old will each be when that happens (in human years)?

Three Solution Methods:

We're going to solve this problem using three different solution methods:

I. Make a table,
II. Make a graph, and
III. Write and solve an equation.

I. Make a Table

Complete this table to see when Joey and Scruffy will have the same age. The first row in the table shows Joey's age when Scruffy was born.

Determining When Scruffy Will Be as Old as Joey		
Year	Joey's Age	Scruffy's Age
0	21	0
1	22	7

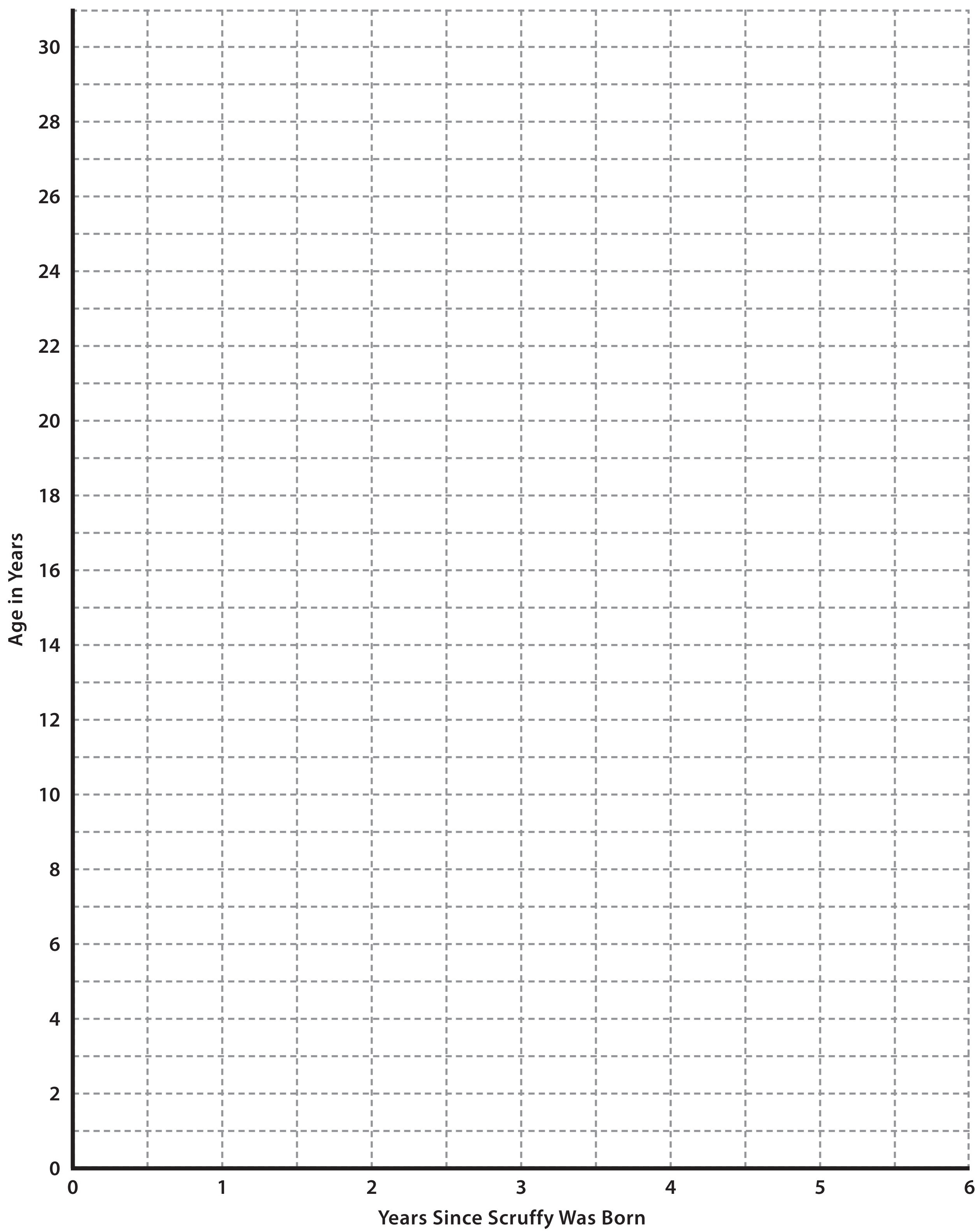
a. Scruffy will be as old as Joey in __________ years.

b. Each will be __________ years old (in human years).

II. Make a Graph

Make a *dual line* graph to plot the ages of Joey and of Scruffy and then examine where the lines meet.

Title of Graph:



a. Scruffy will be as old as Joey in __________ years.

b. Each will be __________ years old (in human years).

III. Write and Solve an Equation

Write an equation to describe the situation.

Hints: Let x = the number of years it will take for Scruffy to be as old as Joey.
So Joey will be $21 + x$ when he and Scruffy are the same age.

a. Scruffy will be as old as Joey in __________ years.

b. Each will be __________ years old (in human years).

Extension

Use any method to solve this problem.

Ben was born on Jan. 31, 1988. Scruffy was born on Feb. 12, 2006. For every year Ben ages, Scruffy ages 7 years. On what date will Scruffy be as old as Ben?

Hint: You may want to convert years to days.

Make No *Bones* about It:
A Forensic Science Investigation*
Teacher's Notes

- **NCTM Standards:** Algebra; Measurement; Number and Operations
- **CCS Standards:** Expressions and Equations; Make sense of problems and persevere in solving them.
- **Mathematical Topics:** Evaluate algebraic expressions; use formulas; solve equations; measure to the nearest millimeter; use exponents
- **Grouping of Students:** Work in pairs or individually

Background

The word *forensic* comes from the Latin *forensic,* meaning "to the forum." The forum was the basis of Roman law and was a place of public discussion and debate pertinent to the law.

In addition to using skeletal remains to estimate a victim's height, forensic scientists also use remains to estimate the age, sex, ancestry, stature, and various unique features of the deceased. Today, advances in technology—especially in the area of DNA testing—has made forensic science much more definitive than ever before.

In this activity lesson, students use formulas to match the bones of victims (of a plane crash) to the identities of missing people. With students working in pairs, one may begin with Victim A and read the length of the bone (humerus, 32.8 cm), while the other enters the data into the appropriate formula:

$$H = 64.977 + 3.144 \cdot h.$$

For Victim A, the result is about 168.1. It appears that Missing Person T (a 19-year-old female who was 168 cm tall) is a match for Victim A. But due to shrinkage of older people, students should also check out others, say, Missing Person Y (who was 155 cm tall). Since Missing Person Y was 55 years old—25 years older than 30—she would have experienced 0.06 × 25, or 1.5 cm, of shrinkage. When 1.5 cm is subtracted from 168.1 cm, the result is 166.6 cm. Since Missing Person Y—and none of the other missing females—was close to 166.6 cm tall, Missing Person T is indeed the likely match.

When students apply the appropriate formula for Victim C, they obtain a height of about 154.3 cm. Here the shrinkage factor plays a key role (since each victim near that height is older than 30, namely 55). Although Missing Person Y's height (155 cm) is closer to 154.3 than Missing Person V's height (153 cm), we need to subtract 0.06 × 25, or 1.5 cm, from 154.3 cm. Since 154.3 cm – 1.5 cm = 152.8 cm, the bones likely are those of Missing Person V (a 55-year-old female who was 153 cm tall).

*The source for the skeletal formulas used in this activity lesson is George Knill, "Mathematics in Forensic Science," *Mathematics Teacher* (February 1981), pp. 31–32.

You may want pairs of students to explain to the class how they used the formulas to make the identifications.

The context for this activity lesson is of high interest to many students—especially due to some popular television programs that deal with this subject matter. One is *NUMB3RS,* a CBS program about the mathematical-genius brother of an FBI agent who uses mathematics to help the FBI solve cases.

Mathematical Humor

Did You Know? Your funny bone is not a bone. Rather, it is the ulnar nerve located near your humerous. (And there is nothing humorous about it.)

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The following story is adapted from the work of Danish humorist Victor Borge (1909–2000).

A 4-engine airplane is traveling at 480 mi/h on a 3-hour, 1,440-mile trip. All of a sudden the pilot makes an announcement.

Pilot: "One of our engines has just given out, but don't worry. This plane can be safely flown with just 3 engines. Our speed is being reduced to 360 mi/h, so the trip will take about a third longer than expected—about 4 hours in all."

A few minutes later, the pilot makes another announcement.

Pilot: "I regret to inform you that another engine just gave out. But we can safely fly with just 2 engines. We will travel at 240 mi/h, and it will now take twice as long as originally planned, or 6 hours. But don't worry about anything."

But then the pilot makes yet another announcement.

Pilot: "A third engine has just given out. We're now safely flying with just 1 engine. We're traveling at 120 mi/h, and the trip will now take 12 hours."

An alarmed passenger then whispers to another passenger, "If we have any more trouble, we're likely to be up here all night!"

Solutions

1a. Victim A—Missing Person T
b. Victim B—Missing Person Z
c. Victim C—Missing Person V
d. Victim D—Missing Person W
e. Victim E—Missing Person X

2. Answers will vary. Suppose a 14-year-old male student's height is 162.6 cm. The length of the student's femur is found as follows, where H is the student's height and f is the length of the student's femur:

$$H = 69.089 + 2.238 \bullet f$$

Replace H with 162.6.

$$162.6 = 69.089 + 2.238 \bullet f$$

Subtract 69.089 from both sides of the equation.

$$93.511 = 2.238 \bullet f$$

Divide both sides of the equation by 2.238.

$$41.8 \approx f$$

Round to the nearest tenth.

The boy's femur is about 41.8 cm long.

Extension

The Extension addresses the need for drivers and passengers to be cognizant of how the stopping distance of a car greatly increases as the speed of the car increases. The formula used is one of many used by investigators. It should be noted that the formula is a simplification of the variables associated with stopping a car. Factors not addressed by the formula include the condition of brakes, the pressure applied to the brake pedal, the condition and type of tires, the type of surface, the temperature, moisture on the surface, and more.

Answers:

1a. 272.25 ft   b. 64 ft   c. 496 ft

2a. about 8 times as great

b. Answers will vary. Doubling the speed far more than doubles the stopping distance.

As a further extension, you may want students to estimate the speed of a car based on a car's skid marks by using the following formula, where s is the speed of the car in mi/h:

$$\text{Length of skid marks} = 0.05s^2$$

For example, suppose a car's skid marks are measured to be 180 feet long. Using the formula, and dividing both sides of the equation by 0.05, you obtain $3{,}600 = s^2$. Students could then use any combination of mental math, trial-and-error, and a calculator to determine that the speed was 60 mi/h. Have students compute the speed of a car where the skid marks measure 320 ft in length (80 mi/h).

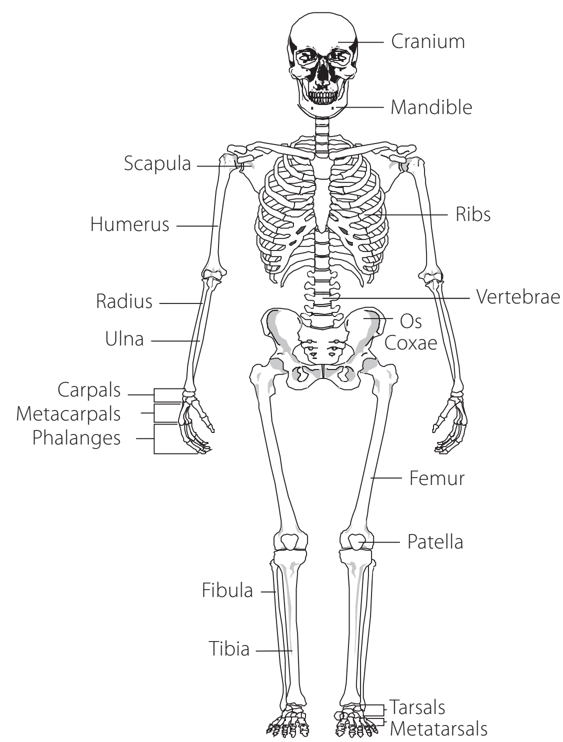
Name____ Date ____________________

Make No *Bones* about It

Materials: centimeter tape measure

Forensic science is the study of science as it applies to criminal and civil investigations. When the bones of a victim are discovered at a crime scene, investigators need to determine the victim's identity. One way to help identify the victim is to use the bones to estimate the victim's height when he or she was living. Generally, the longest bones in the body are the best ones to use.

The formulas provided in the table below can be used to estimate the height *(H)* of a victim based on the lengths of the femur *(f)*, tibia *(t)*, humerus *(h)*, or radius *(r)*. All measurements are given in centimeters.



Males	Females
$H = 69.089 + 2.238 \cdot f$	$H = 61.412 + 2.317 \cdot f$
$H = 81.688 + 2.392 \cdot t$	$H = 72.572 + 2.533 \cdot t$
$H = 73.570 + 2.970 \cdot h$	$H = 64.977 + 3.144 \cdot h$
$H = 80.405 + 3.650 \cdot r$	$H = 73.502 + 3.876 \cdot r$

If the age of the victim is known, an additional calculation needs to be made. In general, after the age of 30, a person's height decreases at the rate of about 0.06 cm per year. So, for a 45-year-old person, if the above formula suggests that he or she is 154.5 cm tall, you would have to subtract 15 years' worth of "shrinkage." This would amount to 15 × 0.06, or 0.9 cm. The bones would suggest that the person currently is 154.5 – 0.9, or 153.6 cm tall. (Note that although the person shrinks, the bones do not.)

A Forensic Investigation

As a result of a tragic plane crash, the remains of five unidentified victims are recovered at the crash site. However, eight people who were on the plane are reported missing. Your task, as a forensics investigator, is to do a preliminary matching of each victim's bones to the identity of a missing person. Note that such a matching will only *suggest* that the bones could be those of the missing person.

Questions

1. Draw a line to match each victim found at the crash site with a reported missing person. Three missing people will not be matched. (When you match victims to missing people, be sure to take the ages of the missing people into account.)

Victim Found at Crash Site (Includes sex of victim found at crash site and length of one of the bones)	**Reported Missing Person** (Includes sex, approximate current height, and current age of missing person)
a. **Victim A (female)** length of humerus: 32.8 cm	**Missing Person S** female, 183 cm tall, 24 years old
b. **Victim B (male)** length of tibia: 43.5 cm	**Missing Person T** female, 168 cm tall, 19 years old
c. **Victim C (female)** length of femur: 40.1 cm	**Missing Person U** male, 177 cm, 25 years old
d. **Victim D (male)** length of femur: 46.5 cm	**Missing Person V** female, 153 cm tall, 55 years old
e. **Victim E (female)** length of radius: 26.5 cm	**Missing Person W** male, 171 cm, 70 years old
	Missing Person X female, 176 cm, 20 years old
	Missing Person Y female, 155 cm, 55 years old
	Missing Person Z male, 186 cm, 28 years old

2. Use a centimeter tape measure to measure your height to the nearest millimeter. Then use the appropriate formula on page 139 and solve for *f, t, h,* or *r* to estimate the length of each of these bones in your body to the nearest tenth of a centimeter.

a. femur ____________ b. tibia ____________

c. humerus _________ d. radius ___________

Extension

Stopping Distance of a Car—Something You *Auto* Know!

Al Lert is traveling in his car at 55 mi/h. All of a sudden he sees an accident ahead of him—so he slams on his brakes. Al is an alert driver, and the road is dry and clear. Still, how far do you think his car traveled from the time he spotted the accident to the time the car skidded to a halt?

The formula below can be used to find the Total Stopping Distance of a car in dry, clear conditions. The stopping distances are much greater in wet, unclear conditions. Trucks require much more stopping distance. Investigators can measure the skid marks left by a car and use a formula such as this to estimate the car's speed. In the formula, s = speed in mi/h.

Perception plus Reaction Distance (in ft)	+	Braking (skidding) Distance (in ft)	=	Total Stopping Distance (in ft)
2.2 • *s*	**+**	**0.05 • s^2**	**=**	***D***

1. Use the formula to compute the Total Stopping Distance for each speed.

a. 55 mi/h ________ b. 20 mi/h ________ c. 80 mi/h ________

(For reference, the length of a football field from goal line to goal line is 300 ft.)

2a. About how many times as great is the total stopping distance of a car traveling at 80 mi/h than a car traveling at 20 mi/h? __________

b. How would you describe the relationship between the speed of a car and how long it takes for it to stop?

____

____

____

Algebra Discovery Lesson 1:
Using an Area Model to Square a Binomial

Teacher's Notes

- **NCTM Standards:** Algebra, Geometry, Number and Operations
- **CCS Standards:** Algebra—Seeing Structure in Expressions; Geometry; Look for and express regularity in repeated reasoning.
- **Mathematical Topics:** Square a binomial; use models to find the area of geometric figures based on the dimensions of a rectangle and partial products; look for a pattern to make a generalization; make a connection to a mental math strategy for multiplying two numbers
- **Grouping of Students:** Work in small groups, in pairs, or individually

Background

This is the first of three activity lessons that employ algebra tiles in a guided discovery format. Have students cut out the algebra tiles on pages 149 and 150 and keep them for use for these lessons. If plastic algebra tiles are available, you may want to use them instead of the paper tiles.

Algebra Discovery Lesson 1 leads students to discover a rule for squaring a binomial of the form $(x + a)^2$, where a is a positive integer. Students use algebra tiles and examine diagrams to build squares with dimensions $x + 1$ on a side, $x + 2$ on a side, and so on. This work aids them in visualizing the partial products that result when a binomial is squared. Students observe how the tiles used to create a large square (the dimensions of which are terms in the binomial) represent the terms of the binomial expansion. Thus, the tiles enable students to make a strong connection between geometry and algebra (often referred to as geometric algebra).

In problem 6, students build a table to show the results of the various area models to enable them to efficiently analyze results across a number of exercises. Students then look for a pattern to formulate a rule for squaring a binomial. Students test their rule with numerical examples—making a key connection between arithmetic and algebra—and then apply the rule to algebraic binomials. The connection to arithmetic (highlighted in problem 8 and the Extension) suggests a way to apply the discovered pattern to create a mental math strategy for multiplication.

Although some mathematics programs use algebra tiles to represent negative quantities, the author subscribes to the belief that working with the concept of "negative area" can be counter-intuitive for many students. As such, the algebra tiles used in these three lessons represent positive quantities only. Once a pattern is established for positive quantities, students may extend the pattern to negative quantities (see Extension).

Following are some of the advantages for using algebra tiles as a vehicle for discovering relationships involved with multiplying and factoring polynomials.

- The "concrete–pictorial–symbolic" sequence enables teachers to reach a broader group of students. Many students have difficulty in establishing a frame of reference for understanding concepts when they are just exposed to abstract manipulation. With algebra tiles, "algebra for all" is attainable because students are able to visualize abstract concepts.
- An *area* model to represent multiplication is readily understood by algebra students. While not all algebraic concepts can be easily modeled with manipulatives, algebra tiles are a natural vehicle for demonstrating basic operations with polynomials.
- Being able to understand the *meaning* of a mathematics concept, as opposed to merely performing the computation, is essential to learning and achievement. Research shows that modeling and visualization promote increased understanding (rather than concentrating on memorization and the abstract).*

*Lawson, Dene R., "The Problem, The Issues that Speak to Change," In *Algebra for Everyone,* edited by Edgar L. Edwards, Jr., pp. 1–6, Reston, VA: National Council of Teachers of Mathematics, 1990.

Mathematical Humor

The discussion of concrete versus abstract brings to mind a story about two teachers, Mr. Jones and Mrs. Rojas. One day the two teachers were walking in the schoolyard when Mr. Jones said, "I would never shout when disciplining a disobedient student. I believe the best approach is being calm at all times." As they continued walking, they observed a student making fingerprints in a freshly laid sidewalk. Mr. Jones immediately screamed, "Young man, get out of that cement immediately!" Later in the day, Mrs. Rojas expressed dismay to Mr. Jones.

Mrs. Rojas: I thought you don't believe in shouting when disciplining a student.

Mr. Jones: Only in the *abstract*, NOT in the *concrete*.

Moral of the story: Teachers should spend more time delving in the concrete (rather than just in the abstract). This will *pave* the way for students to *cement foundational* concepts—with the result being a lasting *impression*.

Solutions

1. There are 5 possible squares.
   They are formed by squares with sides $x + 1$, $x + 2$, $x + 3$, $x + 4$, and $x + 5$.
2. $x^2 + 2x + 1$
3. <u>1</u> x^2 tile, <u>4</u> x tiles, <u>4</u> unit tiles

   $x^2 + 4x + 4$

   $x + 2$ by $x + 2$

   $(x + 2)(x + 2)$

   $x^2 + 4x + 4$
4. <u>1</u> x^2 tile, <u>6</u> x tiles, <u>9</u> unit tiles

   $x^2 + 6x + 9$

   $x + 3$ by $x + 3$

   $(x + 3)(x + 3)$

   $x^2 + 6x + 9$
5. <u>1</u> x^2 tile, <u>8</u> x tiles, <u>16</u> unit tiles

   $x^2 + 8x + 16$

   $x^2 + 8x + 16 = (x + 4)^2$

6a. $(x + 1)^2 = x^2 + \mathbf{2x + 1}$

b. $(\mathbf{x + 2})^2 = x^2 + 4x + 4$

c. $(\mathbf{x + 3})^2 = x^2 + 6x + 9$

d. $(x + 4)^2 = \mathbf{x^2 + 8x + 16}$

7. To square a binomial of the form $(x + a)^2$, record the expansion as the sum of three terms determined as follows: square the x term, double the product of the first and second terms, and square the second term.

8a. *Partial products:*
$(5 + 7)^2 = \underline{25} + \underline{70} + \underline{49} = \underline{144}$
Square of the sum: $(12)^2 = \underline{144}$

b. *Partial products:*
$(20 + 5)^2 = \underline{400} + \underline{200} + \underline{25} = \underline{625}$
Square of the sum: $(25)^2 = \underline{625}$

9a. $x^2 + 10x + 25$ b. $x^2 + 12x + 36$

c. $x^2 + 14x + 49$ d. $x^2 + 2ax + a^2$

Extensions

1. Have students explain how they could use partial products to mentally find $(41)^2$. (Refer then to question 8.)

   Solution

   Answers may vary. Some students may rewrite $(41)^2$ as $(40 + 1)(40 + 1)$ and mentally compute the partial products 1,600 + 40 + 40 + 1 to obtain 1,681.

2. Have students apply their rule for squaring a binomial to binomials involving subtraction, with problems such as these:

   a. $(x - 1)^2$ b. $(x - 2)^2$ c. $(x - 3)^2$ d. $(x - 4)^2$ e. $(x - a)^2$

   Solutions

   a. $x^2 - 2x + 1$ b. $x^2 - 4x + 4$ c. $x^2 - 6x + 9$ d. $x^2 - 8x + 16$ e. $x^2 - 2ax + a^2$

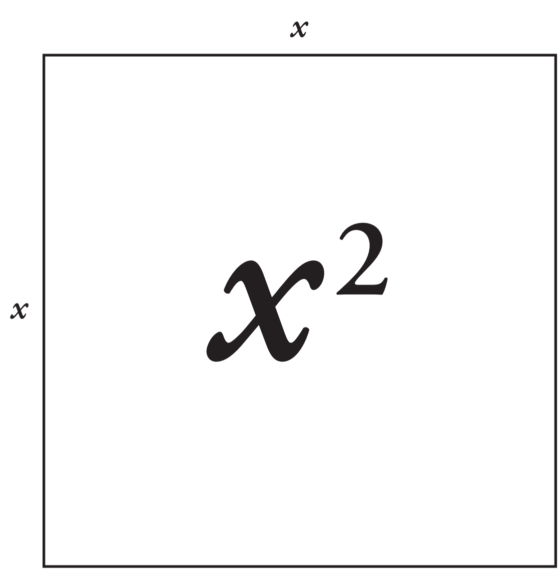
Name____ Date ____________________

Algebra Discovery Lesson 1:
Using an Area Model to Square a Binomial

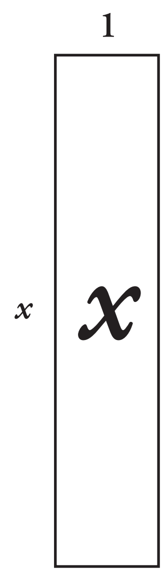
Materials: algebra tiles

In this activity lesson, you will work with rectangles for which some dimensions are not known. The unknown dimensions are represented by the variable x. We call these rectangles *algebra tiles.* Algebra tiles are concrete models for variables and integers that help us explore concepts that are abstract in nature.

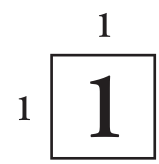
We shall be working with these three kinds of algebra tiles:



The area of this tile in square units is x times x, or x^2. In these lessons, we will simply say that the area is x^2. We call this the x^2 tile.



The area of this tile in square units is 1 times x, or simply x. We call this the x tile.



The area of this tile is 1 times 1, or simply 1. We call this the unit tile.

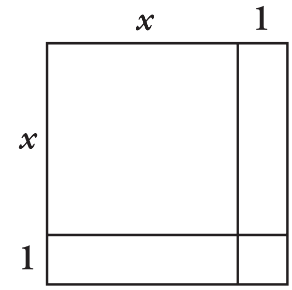
Carefully cut out the algebra tiles found at the end of Lesson 1.

We are going to use the tiles to help us discover a way to square a binomial. A *binomial* is an algebraic expression consisting of two terms. An example of a binomial is $x + 3$. When you multiply a binomial by itself, as in $(x + 3)(x + 3)$, or $(x + 3)^2$, you are *squaring the binomial.*

Questions

1. In this question, the goal is simply to become familiar with the tiles. Use exactly one x^2 tile, up to 10 x tiles, and up to 25 unit tiles. See how many "large" squares (larger than the x^2 tile) you can make. Each "large" square must include one x^2 tile with the other tiles positioned along two sides of the x^2 tile as shown at right.

   We were able to make _______ large squares.



2. Use your tiles to make the large square at the right.
   You used

   1—x^2 tile,
   2—x tiles, and
   1—unit tile.

   Thus, the total area of the large square is $x^2 + 2x + 1$.

The area of a square is given by $s \bullet s = s^2$. Since the dimensions of the large square are $x + 1$ by $x + 1$, we can express the area of that square as $(x + 1)(x + 1) = (x + 1)^2$.

Thus, __________ + __________ + __________ = $(x + 1)^2$.

3. Use tiles to make the large square at the right.
   Complete to show what tiles are used:

   ________ x^2 tile
   ________ x tiles
   ________ unit tiles

   Thus, the total area of the large square is

   __________ + __________ + __________ .

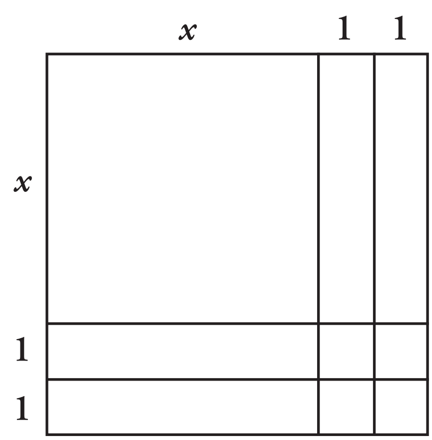
   What are the dimensions of the large square?

   __________ by __________ .

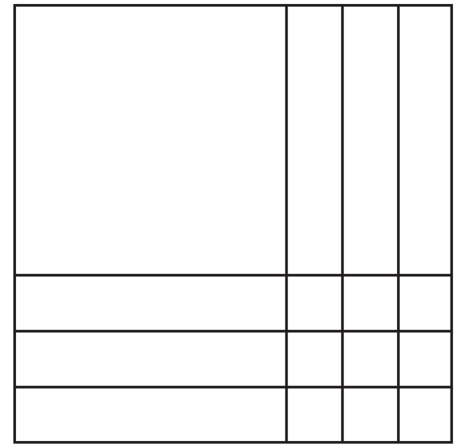
   Use the above dimensions to express the area of the square:

   (+   )(+   ) = $(x + 2)^2$.

   Thus, __________ + __________ + __________ = $(x + 2)^2$.



4. What tiles are used in this square?



________ x^2 tile

________ x tiles

________ unit tiles

Thus, the total area of the large square is

__________ + __________ + __________ .

What are the dimensions of the large square?

__________ by __________ .

Use the above dimensions to express the area of the square:

$(\quad + \quad)(\quad + \quad) = (x + 3)^2$.

Thus, __________ + __________ + __________ = $(x + 3)^2$.

5. Make a large square with dimensions $x + 4$ by $x + 4$.
Complete to show what tiles are used:

________ x^2 tile

________ x tiles

________ unit tiles

Thus, the total area of the large square is

__________ + __________ + __________ .

Thus, __________ + __________ + __________ = $(\quad + \quad)^2$.

6. Use your results from Questions 2–5 to complete this table:

"Squared" Form	Total Area
a. $(x + 1)^2$	x^2 + _________ + _________
b. $(\quad + \quad)^2$	$x^2 + 4x + 4$
c. $(\quad + \quad)^2$	$x^2 + 6x + 9$
d. $(x + 4)^2$	_________ + _________ + _________

7. Closely study the results in the table with question 6 for multiplying a binomial by itself (known as squaring a binomial). Put on your detective hat and see if you can discover a pattern for squaring a binomial (going from the *"Squared" Form* to the *Total Area*).

   Describe your pattern in the space below.

   ____

   ____

   ____

8. Test your pattern with each of these arithmetic examples. In each case, the sum of the *Partial products* determined by your pattern should equal the *Square of the sum.*

   a. *Partial products:* $(5 + 7)^2 = 25 + 70 +$ __________ = __________

   Square of the sum: $(12)^2 =$ __________

   b. *Partial products:* $(20 + 5)^2 =$ __________ + __________ + __________ = __________

   Square of the sum: $(25)^2 =$ __________

9. Now try to complete the rest of the table, below, without using the tiles.

"Squared" Form	**Total Area**
a. $(x + 5)^2$	________ + ________ + ________
b. $(x + 6)^2$	________ + ________ + ________
c. $(x + 7)^2$	________ + ________ + ________
d. $(x + a)^2$	________ + ________ + ________

Algebra Tiles

Cut out the tiles on the following two pages for use in the Algebra Discovery Lessons.

x^2 Tiles

x^2	x^2
x^2	x^2
x^2	x^2

Algebra Tiles (cont.)

x Tiles

x	x	x	x	x	x	x	x	x	x	x
x	x	x	x	x	x	x	x	x	x	x

Unit Tiles

1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1

Algebra Discovery Lesson 2:

Using Algebra Tiles to Multiply Binomials of the Form $(ax + b)(cx + d)$, Where a, b, c, and d Are Whole Numbers

Teacher's Notes

■ **NCTM Standards:**	Algebra, Geometry, Number and Operations
■ **CCS Standards:**	Algebra—Seeing Structure in Expressions; Geometry; Look for and express regularity in repeated reasoning.
■ **Mathematical Topics:**	Multiply binomials; use models to find the area of geometric figures; look for a pattern to make a generalization; explore connections between the FOIL method and a multiplication algorithm
■ **Grouping of Students:**	Work in small groups, in pairs, or individually

Background

This is the second of three activity lessons that employ algebra tiles in a guided discovery format. Students may cut out the algebra tiles that appear on pages 149 and 150.

Algebra Discovery Lesson 2 leads students to discover a rule for multiplying two binomials of the form $(ax + b)$ and $(cx + d)$, where $a, b, c,$ and d are positive integers. Students use algebra tiles and examine diagrams to build rectangles to aid them in visualizing the partial products that result when two binomials are multiplied. Students observe how the tiles used to create a large rectangle (whose length and width match the binomials) represent the terms of the product of the binomials. By "combining like tiles"—which is akin to combining like terms—students are able to build the polynomial for the product of the two binomials. Thus, the tiles enable students to make a strong connection between geometry and algebra.

The most common error students make when multiplying two binomials is to omit recording the two middle terms. As students work through problems 1–3, you may want them to do an "arithmetic check" to verify that the expanded form does indeed equal the binomial form. Students may do this by replacing the variable with a numerical value as shown below for question 1.

Let's replace x with, say, 5.

$$(x + 3)(2x + 1) = 2x^2 + 7x + 3$$

$$(5 + 3)(10 + 1) \stackrel{?}{=} 2(25) + 35 + 3$$

$$(8)(11) \stackrel{?}{=} 50 + 38$$

$$88 = 88$$

In question 4, students examine a generalized area model to help them find a pattern to make a generalization for multiplying two binomials. The *FOIL* method is then introduced as a mnemonic device to help students remember their generalization. Students then apply their generalization in multiplying algebraic binomials.

In questions 7–8, students test the FOIL method by verifying that 37×26 and $(30 + 7)(20 + 6)$ name the same number, and that 46×85 and $(40 + 6)(80 + 5)$ name the same number. This work makes a key connection between arithmetic and algebra. In particular, it shows that the partial products obtained when multiplying two binomials are the same as the partial products obtained when using a standard multiplication algorithm.

Mathematical Humor

Teacher: Expand $(a + b)(c + d)$.

Student: $(a + b)(c + d)$ . . . which can be further expanded as $(a + b)(c + d)$ . . . which can even be further expanded as $(a + b)(c + d)$.

Note: Before computers, it was not easy to do a problem like this.

Solutions

1. $(2x + 1)\ (x + 3)$
   $7x$
   $3;\ 3$
   $2x^2 + 7x + 3$
2. $(3x + 2)\ (x + 3)$
   $3;\ 3x^2$
   $11;\ 11x$
   $6;\ 6$
   $3x^2 + 11x + 6$

3a. $6x^2 + 20x + 6$
 b. $4x^2 + 20x + 21$

4. $bc + bd$
   $ac + ad + bc + bd$
5. multiplied

6a. $x^2 + 8x + 15$
 b. $3x^2 + 23x + 14$
 c. $6x^2 + 22x + 20$
 d. $8ac + 12ad + 12bc + 18bd$

7. $600 + 180 + 140 + 42 = 962$
   The partial products obtained using a standard multiplication algorithm are the same as those obtained by using the FOIL method.
8. $46 \times 85 = (40 + 6)(80 + 5) =$
   $3{,}200 + 200 + 480 + 30 = 3{,}910$

Extensions

1. Have students use a graphing calculator to verify that the graph of the expanded form is identical to the graph of the binomial form. For example, students should graph both $y = 2x^2 + 7x + 3$ and $y = (x + 3)(2x + 1)$ to verify that the graph of each equation is the same parabola.
2. Have students apply their rule for multiplying two binomials to include binomials involving subtraction with problems such as these:
   a. $(x - 1)(x - 2)$
   b. $(x + 2)(x - 3)$
   c. $(x - 3)(x + 3)$
   d. $(2x - 1)(x + 2)$
   e. $(3x + a)(2x - a)$

Solutions

a. $x^2 - 3x + 2$
b. $x^2 - x - 6$
c. $x^2 - 9$
d. $2x^2 + 3x - 2$
e. $6x^2 - ax - a^2$

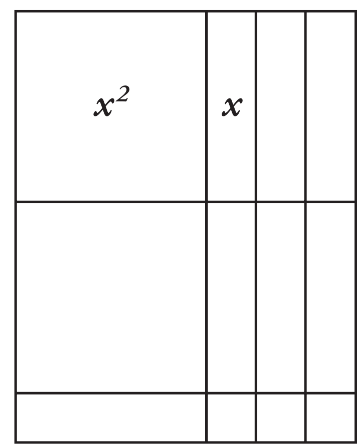
Name____ Date __________________

Algebra Discovery Lesson 2:
Using Algebra Tiles to Multiply Binomials

Materials: algebra tiles

In the previous lesson, we discovered a pattern for multiplying binomials of the form $(x + a)$ and $(x + a)$. In this lesson we will discover how to find the product of *any* two binomials of the form $(ax + b)$ and $(cx + d)$, where *a, b, c,* and *d* are whole numbers.

Questions



1. The area of the large rectangle at right can be found by finding the product of what two binomials?

   (+   )(+   )

   Find the area of each smaller rectangle, and write the area inside the rectangle as shown below.

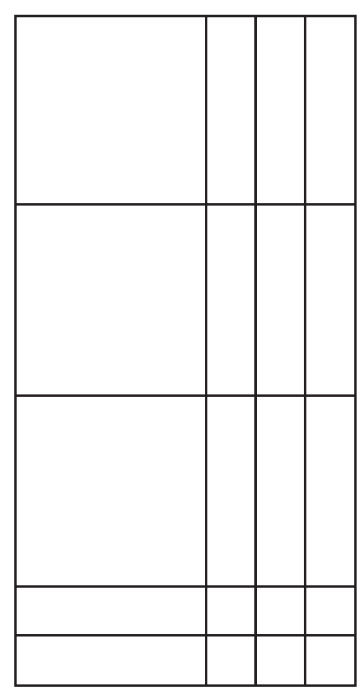
   Since 2 x^2 tiles are used, the area for that portion is $2x^2$.

   Since 7 x tiles are used, the area for that portion is __________ .

   Since __________ unit tiles are used, the area for that portion is __________ .

   Since the area of the large rectangle is the sum of the areas of the smaller rectangles, $(2x + 1)(x + 3) =$

   __________ + __________ + __________



2. The area of the large rectangle at right can be found by finding the product of what two binomials?

   (+   )(+   )

   Since __________ x^2 tiles are used, the area for that portion is __________ .

   Since __________ x tiles are used, the area for that portion is __________ .

   Since __________ unit tiles are used, the area for that portion is __________ .

   Thus, $(3x + 2)(x + 3) =$

   __________ + __________ + __________

3. Find the following products. You may use your tiles to help you. Be sure to *combine like terms* (combine all x^2 tiles together, combine all x tiles together, and combine all unit tiles together).

   a. $(3x + 1)(2x + 6)$ = __________ + __________ + __________

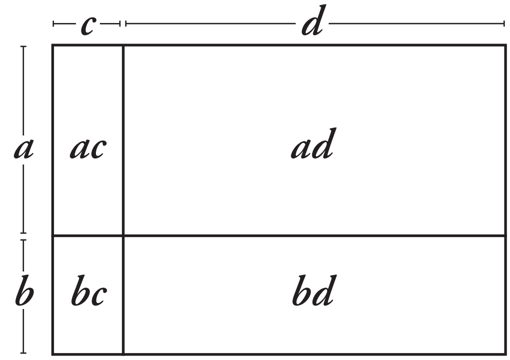
   b. $(2x + 3)(2x + 7)$ = __________ + __________ + __________

4. The area of the large rectangle is found by finding the sum of the areas of each of the smaller rectangles. Thus, the area of the large rectangle is equal to

   $ac + ad$ + __________ + __________

   Thus, $(a + b)(c + d)$ =

   ________ + ________ + ________ + ________



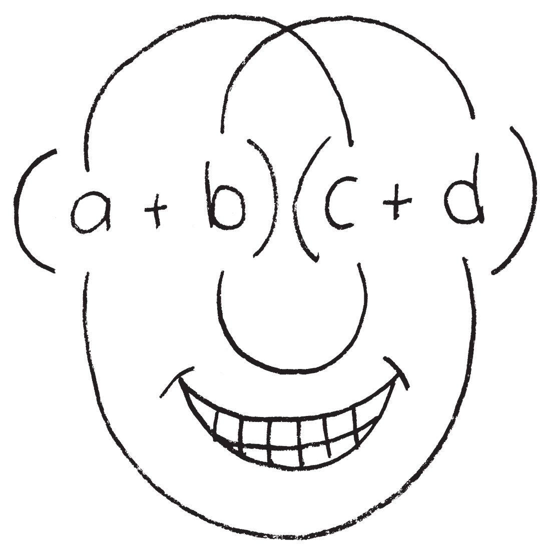
5. Study the above example to discover a pattern for multiplying two binomials. Then complete this statement:

   To find the product of two binomials, each term of the first binomial is ______________ by each term of the second binomial.

The figure at the right should help you remember to multiply each term of the first binomial by each term of the second binomial. The method shown by the drawing is known as the FOIL method. (By drawing in a mouth, the figure looks like a head—thus giving us a "round-about connection" between math and art.)

According to the FOIL method, to multiply two binomials, multiply the . . .

- **F**irst terms (a and c), the
- **O**uter terms (a and d), the
- **I**nner terms (b and c), and the
- **L**ast terms (b and d).



Mr. Kent B. Foiled

6. Find each product. Be sure to combine like terms.

   a. $(x + 3)(x + 5) =$ ____________________________

   b. $(3x + 2)(x + 7) =$ ___________________________

   c. $(2x + 4)(3x + 5) =$ __________________________

   d. $(2a + 3b)(4c + 6d) =$ ________________________

7. Let's use the algebraic pattern we just discovered to help us do the following arithmetic multiplication problem: 37×26.

   First rewrite 37×26 as a product of two binomials, using the largest possible multiple of 10 in each term: $(30 + 7)\ (20 + 6)$.

   Then multiply the binomials, using the *FOIL* method: $(30 + 7)\ (20 + 6) =$

   600 + __________ + __________ + __________ = __________

   Examine the partial products in a standard multiplication algorithm shown at the right.

   How do the partial products in the algorithm at the right compare to the partial products obtained by using the FOIL method?

   ____

   ____

37	
× 26	
42	⇦ 6 × 7
180	⇦ 6 × 30
140	⇦ 20 × 7
600	⇦ 20 × 30
962	

8. Use the FOIL method to find this product: 46×85.

   $46 \times 85 = (\quad + \quad)(\quad + \quad) =$

   __________ + __________ + __________ + __________ = __________

Algebra Discovery Lesson 3:
Using Algebra Tiles to Factor Trinomials of the Form $x^2 + bx + c$, Where b and c Are Whole Numbers
Teacher's Notes

- **NCTM Standards:** Algebra, Geometry
- **CCS Standards:** Algebra—Seeing Structure in Expressions; Geometry; Look for and express regularity in repeated reasoning.
- **Mathematical Topics:** Factor a quadratic trinomial by using an area model; use the strategy of trial-and-error to make a generalization
- **Grouping of Students:** Work in small groups, in pairs, or individually

Background

This is the third of three activity lessons that employ algebra tiles in a guided discovery format. Students may cut out the algebra tiles that appear on pages 149 and 150.

Algebra Discovery Lesson 3 leads students to discover a method for factoring a trinomial of the form $x^2 + bx + c$, where b and c are positive integers. This type of polynomial is known as a *quadratic trinomial* because it is a polynomial of degree 2 with three terms. The degree of a polynomial is the highest degree of any of its terms (after the polynomial has been simplified). The degree of a term is the sum of the term's exponents.

Algebra Discovery Lesson 3 begins with a review of the concept of factoring a number. A connection is then made to algebra in showing that writing a trinomial as a product of two binomials is the same ideas as writing 78 as 2 × 39. Explain that factoring a polynomial is the reverse process of using the FOIL method to multiply two binomials. Thus, you could say that when you factor, you are *unfoiling* the polynomial.

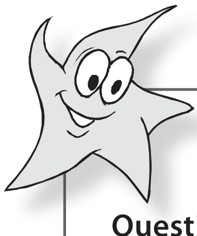
In questions 1–3, students use algebra tiles and examine diagrams to build rectangles to aid them in visualizing the partial products that result when two binomials are multiplied. Students observe how the tiles used to create a large rectangle (whose length and width match the binomials) represent the terms of the product of the binomials. This should lead them to conclude that the dimensions of the large rectangle provide the factored form for the trinomial.

In questions 4–6, students use trial-and-error to build rectangles based on the given trinomials without the aid of the diagrams. In question 7, students build a table to display the results from questions 1–6. They then use this table to help them discover a pattern for factoring a trinomial.

As students work through questions 1–6, you may want them to do an "arithmetic check" to verify that the factored form does indeed equal the trinomial form. Students may do this by replacing the variable with a numerical value as shown below for question 1.

Let's replace x with, say, 6.

$$\begin{aligned} x^2 + 2x + 1 &= (x + 1)(x + 1) \\ 36 + 12 + 1 &\stackrel{?}{=} (6 + 1)(6 + 1) \\ 49 &= 7 \times 7, \text{ or } 49 \end{aligned}$$



Mathematical Humor

Question: What do you tell your parrot when you want her to go on a diet?
Answer: Poly-no-meal.

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

Question: What do you say when your parrot flies away?
Answer: Poly-gone.

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

Teacher: Find x in the equation $x^2 + 2x + 1 = 0$.
Student: Here it is.

$$x^2 + 2x + 1 = 0$$

Solutions

1. $x^2 + 2x + 1$
   $x + 1$ by $x + 1$
2. $x^2 + 4x + 4$
   $x + 2$ by $x + 2$
   $x^2 + 4x + 4$
3. $x + 2$ by $x + 1$
   $x^2 + 3x + 2 = (x + 2)(x + 1)$
   $x^2 + 3x + 2$
4. $x + 3$ by $x + 2$
   $(x + 3)(x + 2)$
5. $x + 4$ by $x + 1$
   $(x + 4)(x + 4)$

6a. $(x + 4)(x + 2)$
b. $(x + 5)(x + 1)$
c. $(x + 5)(x + 2)$
d. $(x + 1)(x + 6)$
e. $(x + 4)(x + 3)$
f. $(x + 7)(x + 1)$
g. $(x + 6)(x + 2)$

7a. $x^2 + 2x + 1 = \mathbf{(x + 1)(x + 1)} = \mathbf{(x + 1)^2}$
b. $x^2 + 4x + 4 = \mathbf{(x + 2)(x + 2)} = \mathbf{(x + 2)^2}$
c. $x^2 + 3x + 2 = \mathbf{(x + 2)(x + 1)}$
d. $\mathbf{x^2 + 5x + 6} = (x + 3)(x + 2)$
e. $\mathbf{x^2 + 5x + 4} = (x + 4)(x + 1)$
f. $x^2 + 6x + 8 = \mathbf{(x + 2)(x + 4)}$
g. $\mathbf{x^2 + 6x + 5} = (x + 5)(x + 1)$
h. $\mathbf{x^2 + 7x + 10} = (x + 5)(x + 2)$
i. $x^2 + 7x + 6 = \mathbf{(x + 6)(x + 1)}$
j. $x^2 + 7x + 12 = \mathbf{(x + 4)(x + 3)}$
k. $x^2 + 8x + 7 = \mathbf{(x + 7)(x + 1)}$
l. $x^2 + 8x + 12 = \mathbf{(x + 2)(x + 6)}$

8. The product of the numerical (or last) terms of the factors is equal to the numerical term of the trinomial. The sum of the numerical terms of the factors is equal to the coefficient of the x term of the trinomial. The first term of each factor is x.

Extensions

1. Have students use a graphing calculator to verify that the graph of the factored form is identical to the graph of its expanded form. For example, students should graph both $y = (x + 1)(x + 1)$ and $y = x^2 + 2x + 1$ to verify that the graph of each equation is the same parabola.

2. Have students use factoring to simplify the fraction shown below, where $x \neq -1$ and $x \neq -3$. Explain that students should first factor the numerator and the denominator. Then they should simplify the expression by "canceling" common factors.

$$\frac{x^2 + 2x + 1}{x^2 + 4x + 3}$$

Have students do an "arithmetic check" to verify that the simplified form is equivalent to the original expression.

Solution

Factoring the numerator and denominator and then "canceling" yields the following:

$$\frac{(x+1)(x+1)}{(x+1)(x+3)} = \frac{\cancel{(x+1)}(x+1)}{\cancel{(x+1)}(x+3)} = \frac{(x+1)}{(x+3)}$$

Let's verify the results by replacing the variable with, say, 4.

$$\frac{(4+1)}{(4+3)} \stackrel{?}{=} \frac{4^2 + 2(4) + 1}{4^2 + 16 + 3}$$

$$\frac{5}{7} \stackrel{?}{=} \frac{16 + 8 + 1}{16 + 16 + 3}$$

$$\frac{5}{7} \stackrel{?}{=} \frac{25}{35}$$

$$\frac{5}{7} = \frac{5}{7}$$

3. Have students apply their rule for factoring to include trinomials involving subtraction with problems such as these:

a. $x^2 - x - 6$

b. $x^2 + 4x - 5$

c. $x^2 - 5x - 6$

d. $x^2 - 3x + 2$

e. $x^2 - 8x + 12$

Solutions

a. $(x - 3)(x + 2)$

b. $(x + 5)(x - 1)$

c. $(x - 6)(x + 1)$

d. $(x - 2)(x - 1)$

e. $(x - 6)(x - 2)$

Name____ Date ___________________

Algebra Discovery Lesson 3:
Using Algebra Tiles to Factor Trinomials

Materials: algebra tiles

When you multiply two numbers (say, 2 and 39), the result is a single number (78). *Factoring a number* is the reverse process. Here we begin with a number (78) and write it as a *product* of two or more numbers. We could *factor* 78 as 2×39.

A *trinomial* is a polynomial with three terms. An example of a trinomial is $x^2 + 8x + 15$. We have learned that $(x + 3)(x + 5) = x^2 + 8x + 15$.

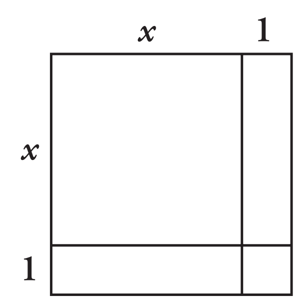
- Thus we can multiply $(x + 3)$ and $(x + 5)$ to obtain $x^2 + 8x + 15$.

Factoring is the reverse process. Factoring a polynomial is the process of expressing the polynomial as a product of two or more expressions.

- When we express $x^2 + 8x + 15$ as $(x + 3)(x + 5)$, we have *factored* $x^2 + 8x + 15$.

Factoring trinomials is not easy. Often we use trial-and-error to help us find the factors. Algebra tiles can make our work easier. Since Algebra tiles show area relationships, the length and width of a "large" rectangle can sometimes give us the desired factors.

Questions



1. Use tiles to make the large rectangle at the right. Complete to show what tiles are used:

   The total area of the large rectangle is

   __________ + __________ + __________ .

   What are the dimensions of the large rectangle?

   __________ by __________ .

   The area of the large rectangle (in this case it is a special rectangle—namely a square) may thus be expressed as $(x + 1)(x + 1)$. Thus, $x^2 + 2x + 1 = (x + 1)(x + 1)$.

   We say that $(x + 1)(x + 1)$, or $(x + 1)^2$, is the factored form for $x^2 + 2x + 1$.

2. Use tiles to make this large rectangle:
   The total area of the large rectangle is

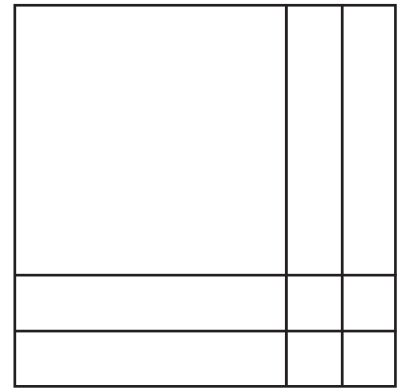
   __________ + __________ + __________ .

   What are the dimensions of the large rectangle?

   __________ by __________ .

   Thus, $x^2 + 4x + 4 = (x + 2)(x + 2)$.
   We say that $(x + 2)^2$ is the factored form for

   __________ + __________ + __________ .



3. Use the following blocks to make a large rectangle:
   1 — x^2 tile,
   3 — x tiles, and
   2 — unit tiles.
   Give the dimensions of the large rectangle you made: __________ by __________ .

   Thus, __________ + __________ + __________ = (+   )(+   ).

   $(x + 2)(x + 1)$ is the factored form for __________ + __________ + __________

4. Factor the following trinomial: $x^2 + 5x + 6$.
   Hint: Form a large rectangle with these tiles:
   1 — x^2 tile,
   5 — x tiles, and
   6 — unit tiles.
   Give the dimensions of the large rectangle you made: __________ by __________ .
   Thus, (+   )(+   ) is the factored form for $x^2 + 5x + 6$.

5. Factor the following trinomial: $x^2 + 5x + 4$.
   Give the dimensions of the large rectangle you made: __________ by __________ .
   Thus, (+   )(+   ) is the factored form for $x^2 + 5x + 4$.

6. For each of the following trinomials, use tiles to make a large rectangle.
   Then give the factored form.
   a. $x^2 + 6x + 8$ = ______________________________
   b. $x^2 + 6x + 5$ = ______________________________
   c. $x^2 + 7x + 10$ = ______________________________
   d. $x^2 + 7x + 6$ = ______________________________
   e. $x^2 + 7x + 12$ = ______________________________
   f. $x^2 + 8x + 7$ = ______________________________
   g. $x^2 + 8x + 12$ = ______________________________

7. Record your results from questions 1–6 in the table below.

Trinomial	Factored Form
a. $x^2 + 2x + 1$	$(\quad + \quad)(\quad + \quad) = (\quad + \quad)^2$
b. $x^2 + 4x + 4$	$(\quad + \quad)(\quad + \quad) = (\quad + \quad)^2$
c. $x^2 + 3x + 2$	$(\quad + \quad)(\quad + \quad)$
d. ________ + ________ + ________	$(x + 3)(x + 2)$
e. ________ + ________ + ________	$(x + 4)(x + 1)$
f. $x^2 + 6x + 8$	
g.	$(x + 5)(x + 1)$
h.	$(x + 5)(x + 2)$
i. $x^2 + 7x + 6$	
j. $x^2 + 7x + 12$	
k. $x^2 + 8x + 7$	
l. $x^2 + 8x + 12$	

8. Study the above table carefully. Put on your detective hat and see if you can discover a pattern for factoring a trinomial.

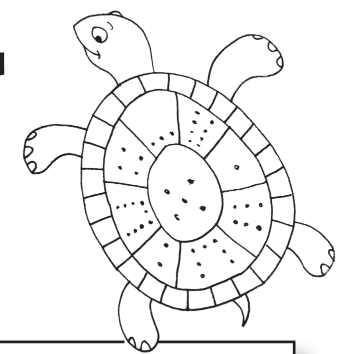
   Describe your pattern in the space below.

5 Focus on Mathematical Recreation

Mathematics has beauties of its own—a symmetry and proportion in its results. When this subject is properly . . . presented, the mental emotion should be that of enjoyment of beauty, not that of repulsion from the ugly and the unpleasant.
—John Wesley Young, American mathematician (1879–1932)

Doing *Sum* Exploring with Magic Squares

Teacher's Notes



- **NCTM Standards:** Number and Operations; Geometry; Data Analysis
- **CCS Standards:** Operations and Algebraic Thinking; Geometry; Look for and make use of structure.
- **Mathematical Topics:** Addition of whole numbers; use rotations (turns) and reflections (flips); identify the mean and the median; find a pattern
- **Grouping of Students:** Work independently or in pairs

Background

Let no one ignorant of geometry enter here.

—Inscription at the door of the academy of Plato, Greek philosopher and educator (427? B.C.–347 B.C.)

This activity lesson integrates addition skills with concepts of geometry. Central to the lesson is the searching and discovery of patterns. Although magic squares can be successfully completed using trial-and-error, students should recognize that applying a pattern is generally more efficient.

The earliest known magic square, recorded around 2800 B.C., is based on a Chinese myth. Fuh-Hi described the *Lo Shu,* or "scroll of the river Lo" in the book *Yih King.* The *Lo Shu* is a 3 × 3 magic square drawn on a turtle's shell. Odd numbers are expressed by white dots (the emblem of heaven). Even numbers are represented by black dots (the emblem of earth). The number 1 represents the beginning of all things; 9 represents completion. The number 5 represents the emperor and China. It is at the center—from where it exerts its influence.

There are a number of excellent Web sites that provide a wealth of information on magic squares. Some of them are listed below.

- **Grogono Magic Squares: A Mini-History of Magic Squares; Grogono Magic Squares Home Page**
  The site includes information on creating magic squares—from 3 × 3 through 13 × 13 magic squares.
  http://www.grogono.com/magic/history.php
  http://www.grogono.com/magic/index.php
- **Patterns and Magic Squares**
  The site includes an interactivity that performs transformations on magic squares.
  http://www3.sympatico.ca/diharper/magic.html
- **Dürer's Magic Square**
  This site shows Albrecht Dürer's 1514 engraving that contains a magic square—the first magic square published in Europe. The site also includes a discussion of the mathematics behind Dürer's magic square that is enhanced by an applet.
  http://www.taliscope.com/Durer_en.html

- **How to Construct Magic Squares**
  (from the Math Forum)
  http://mathforum.org/alejandre/magic.square/adler/adler4.html

Mathematical Humor

"Can you do Addition?" the White Queen asked. "What's one and one and one and one and one and one and one and one and one and one?"

"I don't know," said Alice. "I lost count."

"She can't do Addition," the Red Queen interrupted. "Can you do Subtraction? Take nine from eight."

"Nine from eight I can't, you know," Alice replied very readily: "but—"

"She can't do Subtraction," said the White Queen. "Can you do Division? Divide a loaf by a knife—what's the answer to *that?*"

"I suppose—" Alice was beginning, but the Red Queen answered for her. "Bread-and-butter, of course. Try another Subtraction sum. Take a bone from a dog: what remains?"

Alice considered. "The bone wouldn't remain, of course, if I took it—and the dog wouldn't remain: it would come to bite me—and I'm sure *I* shouldn't remain!"

"Then you think nothing would remain?" said the Red Queen.

"I think that's the answer."

"Wrong, as usual," said the Red Queen: "The dog's temper would remain."

"But I don't see how—"

"Why, look here!" the Red Queen cried. "The dog would lose its temper, wouldn't it?"

"Perhaps it would," Alice replied cautiously.

"Then if the dog went away, its temper would remain!" the Queen exclaimed triumphantly.

—From *Through the Looking Glass* by Charles Lutwidge Dodgson,
under the pen name Lewis Carroll (1832–1898)

Solutions

1.

Original

4	9	2
3	5	7
8	1	6

Rotate the original square clockwise 90°.

8	3	4
1	5	9
6	7	2

Rotate again clockwise 90°.

6	1	8
7	5	3
2	9	4

Rotate again clockwise 90°.

2	7	6
9	5	1
4	3	8

Reflect the first and third columns of the original over the middle column.

2	9	4
7	5	3
6	1	8

Reflect the top and bottom rows of the original over the middle row.

8	1	6
3	5	7
4	9	2

Reflect the 4, 3, & 9 and the 6, 1, & 7 in the original over the diagonal through 8, 5, & 2.

6	7	2
1	5	9
8	3	4

Reflect the 8, 3, & 1 and the 2, 9, & 7 in the original over the diagonal through 4, 5, & 6.

4	3	8
9	5	1
2	7	6

2. The (positive) difference between each pair of numbers opposite the diagonal is 2.
3. The (positive) difference between each pair of numbers opposite the diagonal is 6.
4. Begin with the upper left-hand corner (magic square at right). The numbers wrap down and around consecutively, 1–5, up the middle column. A similar pattern begins with the bottom right-hand corner.

↓ 1	↓ 4	← 3
↓ 2	5	2 ↑
→ 3	↑ 4	↑ 1

5. Underline all answers inside the parentheses that apply:
   5 is the (<u>mean</u>, <u>median</u>, mode) of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9.

6. Sample magic square:

6	11	4
5	7	9
10	3	8

7. Sample magic square:

38	43	36
37	39	41
42	35	40

8 a. 21

b. 117

c. It is 6 greater.

d. It is 102 greater.

e. Sample Answer: You could add the same number to each number in a magic square to create a new magic square. If the number you add is *n,* then the new magic square will have a magic sum that is $3 \bullet n$ more than the original magic sum.

9. Magic squares will vary. Check each student's work.

10.

27	32	25
26	28	30
31	24	29

Extensions

1. The Extension included in the activity lesson is based on discovering a pattern for creating a 5-by-5 magic square. The solutions appear below.

a. Begin in the middle cell in the top row with the first number (in this case, 1). Fill in the cells diagonally upward and to the right with consecutive numbers. There are two basic exceptions.

(1) If you land outside of the square, *on top,* then you get back into the square by going to the bottom of the next column. If you land outside of the square, *on the right,* then shift completely across to the left and go to the first cell in the next row up. Continue with the general pattern. (The cell in the top right-hand corner "wraps" to the bottom cell in the first column on the left, unless, of course, it is occupied.)

(2) If you land in a cell that is already occupied (or *on top* in the far right column), then you must write the number in the cell immediately below the one last filled. Continue with the general pattern.

(The above method was created by Frenchman Antoine de la Loubère in 1688.)

b. The pattern works for any magic square of an *odd order* (3×3, 5×5, 7×7, etc.).

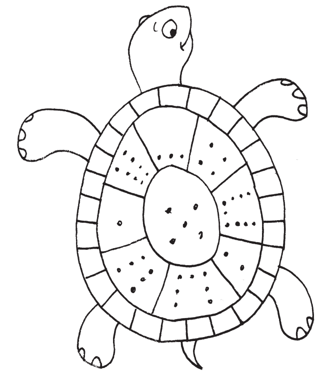
c. A possible answer is shown below. The magic sum is 90.

22	29	6	13	20
28	10	12	19	21
9	11	18	25	27
15	17	24	26	8
16	23	30	7	14

2. The overall activity lesson can be extended by having students solve and/or create magic squares that consist of decimals, fractions, or integers.

Name____ Date ___________________

Doing *Sum* Exploring with Magic Squares



A *magic square* is square array of numbers where the sum of the numbers in each row, column, and diagonal is the same. This sum is called the *magic sum.* The earliest known magic square was recorded around 2800 B.C. It consisted of symbols arranged in a 3-by-3 array drawn on the back of a turtle. This magic square, written with our familiar Hindu-Arabic numerals, is shown at right.

4	9	2
3	5	7
8	1	6

There are eight magic squares that can be formed using the whole numbers 1–9. Beginning with an original magic square, *rotate* (turn) it 90° three times to create three more magic squares. You can *reflect* (flip) various elements of the original to create four more magic squares.

Questions

1. Follow the directions to create seven variations of the original magic square.

Original

4	9	2
3	5	7
8	1	6

Rotate the original square clockwise 90°.

		4
		9
		2

Rotate again clockwise 90°.

Rotate again clockwise 90°.

Reflect the first and third columns of the original over the middle column.

Reflect the top and bottom rows of the original over the middle row.

Reflect the 4, 3, & 9 and the 6, 1, & 7 in the original over the diagonal through 8, 5, & 2.

Reflect the 8, 3, & 1 and the 2, 9, & 7 in the original over the diagonal through 4, 5, & 6.

2. Suppose you draw a diagonal through 8, 5, & 2 in the original square and fold along the diagonal. What symmetrical relationship would you see?

____

3. Suppose you draw a diagonal through 4, 5, & 6 in the original square and fold along the diagonal. What symmetrical relationship would you see?

____

4. The number 5 is in the center of the original square. Find the (positive) difference of each of the other numbers in the original square and 5. Record those differences in their positions in the grid at the right. What patterns do you notice?

1		
	5	

____

____

5. Underline all answers inside the parentheses that apply:

5 is the (mean, median, mode) of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9.

6. Use *patterns* to help you make a magic square with the whole numbers 3–11.

7. Make a magic square using the whole numbers 35–43.

8a. What is the magic sum in #6? ___________ b. in #7? ___________

c. How does the magic sum in #6 compare to the one in #1? _______________________

d. How does the magic sum in #7 compare to the one in #1? _______________________

e. What could you do to each number in a magic square to create a new magic square that has a different magic sum?

____

____

____

9a. Create your own 3 × 3 magic square that has a different magic sum from any of the others in this lesson.

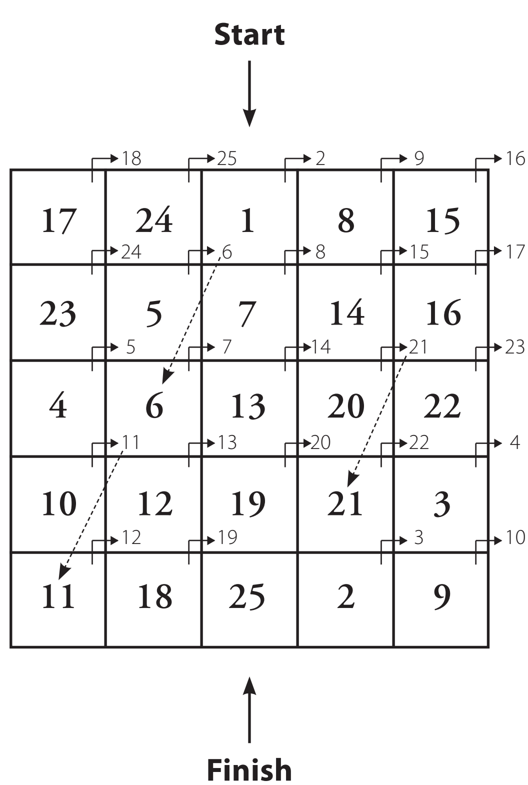
b. What is the magic sum? ___________

10. Complete this magic square.

		25
		30
		29

Extension

The diagram below shows a method for creating a 5 × 5 magic square using the numbers 1–25. Follow the arrows to discover a *pattern.*



a. Describe the pattern.

____

____

____

____

____

b. Test to see if the pattern helps you make a 3×3 magic square.

c. Use the pattern in part a to help you create a 5×5 magic square using the whole numbers 6–30.

What is the magic sum? __________

The Mathematical Treasure Hunt

Teacher's Notes

- **NCTM Standards:** Geometry; Number and Operations; Algebra
- **CCS Standards:** Geometry; Operations and Algebraic Thinking; Make sense of problems and persevere in solving them.
- **Mathematical Topics:** Angle/side relationships in a triangle; use a factor tree; evaluate expressions; use logical reasoning
- **Grouping of Students:** Work individually or in pairs

Background

It is impossible to be a mathematician without being a poet in soul.

—Sophia Kovalevskaya,
Russian mathematician and novelist (1850–1891)

The author's daughter, Jamie, wrote this story with poems as part of a mathematics writing assignment in her eighth-grade mathematics class. Many researchers and mathematics educators have studied the positive effects of having students engage in writing assignments in their mathematics class. According to the NCTM Process Standard, Communication, "Writing is a valuable way of reflecting on and solidifying what one knows" (NCTM, 2000). According to Miller (1991), "A great deal can be learned from students when they are asked to write about their understanding of mathematics." According to Mason (2002), "Students may develop a variety of reasons for writing in mathematics class—to *respond* to a teacher's request for evidence of mathematical understanding, to *report* on what they want the teacher to know, to *reflect* on their own learning, and to *relate* to the teacher as a partner in their learning."

Mathematical Humor

Encourage students to create their own mathematical jokes or riddles. It is quite likely that the following riddles were created by students.

Tell what comes next: 2, 4, 6, 8, . . .
Answer: Who do we appreciate?

What should you do if you want a pet zoid?
Answer: trap—ezoid.

Mathematical Humor

What is $5q + 5q$?
Answer: $10q$. You're welcome.

What do acorns say when they grow up?
Answer: Gee– I'm–a–tree.

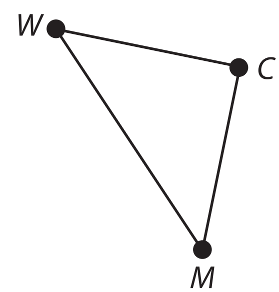
What do you call two lells?
Answer: Pair–a–lell

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

Geometry is just *plane* fun.
"This is a mirror image," *reflected* Juan.
"$8 \times 8 = 64$," Allie said *squarely.*
"A repeating decimal does not end," commented Wanda in her *infinite* wisdom.
Jim *flat out* said, "$A = l \cdot w$." But Eva *solidly* remarked, "$V = l \cdot w \cdot h$."
"This segment has come to an end," David said *pointedly!* And everyone said, "*Dot's* good!"

Solutions

1. By measurement, students will determine that angle C has the greatest measure. (There is a theorem in geometry that states that the angle opposite the longest side of a triangle has the greatest measure.)



2. $135 = 15 \times \underline{9} = 5 \times 3 \times \underline{3} \times \underline{3}$, so the answer is "**O:** 9, 3, 3."
3. We can organize the information into a table. This table shows that Claudia accused Bob of doing it, Bob accused Jack, and Jack denied doing it.

	Claim Made by Each Person		
	Claudia	Bob	Jack
Claudia	"Bob did it."		
Bob		"Jack did it."	
Jack			"I did not do it."

Claudia and Bob cannot both be telling the truth because each is accusing a different person of the "crime." Since exactly *one* person lied, either Claudia or Bob is lying—and Jack is telling the truth. So Jack did not do it. This means Bob is lying and Claudia is telling the truth. Hence, Bob did it.

The second letter in Bob's name is O, so the letter O is written in the Solution Box on page 175.

4. Automobile speeds (for distances given in miles) are given in miles per hour.

So PER is written in the Solution Box.

5. $28 - 7a = 14 + t$ is a true statement when $\boldsymbol{a = 3}$ **and** $\boldsymbol{t = -7}$.

$$\begin{aligned} 28 - 7a &= 14 + t \\ 28 - 7(3) &= 14 + (-7) \\ 28 - 21 &= 7 \\ 7 &= 7 \end{aligned}$$

   So, A and T are written in the Solution Box.

6. A rectangular prism has 12 edges, so the letter E is written in the Solution Box.

Solution Box

Letter	C	O	O	P	E	R	A	T	E
Question	**1**	**2**	**3**	**4**	**4**	**4**	**5**	**5**	**6**

Extension

Ask students to write their own math story, song, and/or poem. You may want to create a grading rubric that takes into account mathematical content; creativity; basic grammar, spelling, and punctuation; overall organization, neatness, and presentation; use of representations (graphs, tables, etc.); and more.

The author's daughter, Jamie, wrote the following song when she was in eighth grade. You may want to gradually build factor trees for 40 and 54 at the board while your students sing the lyrics to the song.

My Factor Tree

Sung to the tune of "My Bonnie Lies Over the Ocean"
Lyrics by Jamie Spangler

My factor tree starts with a number,
I can choose any number I please.
I break it to two of its factors,
And then I continue with ease.

Break down, break down, all the composites you see,
Break down, break down, 'til just primes are left on our tree.

Now let's try the number 40,
2 is a factor you know.
So we're left with 20 and 2,
We can still break down 20—let's go.

Break down, break down, 20 to 4 and 5,
Break down, the composite 4, the prime factors are 2, 2, 2, and 5.

Now let's try the number 54,
Which is equal to 27 times 2.
27 is divisible by 9 and 3,
And I'm sure that you know what we'll do.

Break down, the nine to two threes, so our factors are 2, 3, 3, and 3,
Break down, break down, now only primes remain on our tree!

Name____ Date ___________________

The Mathematical Treasure Hunt

by Jamie Spangler (eighth-grade student)

Materials: protractor

As you read this story, you will be asked to solve a mathematical problem from time to time. Please record your answers, as directed, in the Solution Box that immediately follows Question 6.

There were only three houses in the vast Discovery Woods. The houses were non-collinear, so they formed a triangle. The families living there were the Carings in the east, the Midsts in the south, and the Wantmores in the west. The families were rather different, so they rarely had contact with one another. However, on one fateful day all of that changed.

It was a warm summer night, and everyone was asleep. Suddenly, a masked figure approached the Carings' house and slipped a note in their door. The person then disappeared into the darkness. The commotion awakened the Carings' daughter, Janet. She ran downstairs to find the peculiar note. It said:

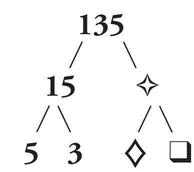
I know this fact will be to your pleasure,
Somewhere in the woods there's a buried treasure.
To help the search along, I will give clues,
That, with math skills, are simple to use.
The Carings' house I'll call point *C*,
Point *M* is what the Midsts' house will be.
I will refer to the Wantmores' house as point *W*,
Now I'll tell you the purpose of the points so it won't trouble you.
The longest side is *MW*, I must say,
Now find the largest angle in any way.

1. Okay reader, it's time to solve your first mathematical problem. Make a diagram of the points, with the longest side being *MW*. Then use a protractor to see which angle has the greatest measure. Write the letter of the vertex of that angle in the Solution Box for Question 1.

Janet was a kind person, so she showed this clue to her family as well as the other families in Discovery Woods. Once they figured out this problem, they were quite content. Although they didn't know what to make of this mystery, they knew they were on the right track.

The next day, Mr. Midst found this note taped to his tree:

This oak tree isn't your average tree,
It's an amazing factor tree, you see.
We'll start with its age, 135,
Here's a tree to factor how long it has been alive.
Find the missing digits, if you can,
But ask for the help of every clan.



2. It's your turn again, reader. Figure out which letter below provides the correct answers for ✧, ◊, and ❑ in that order.

A: 7, 7, 1 E: 14, 3, 3 I: 7, 9, 9 O: 9, 3, 3

Record the correct letter in the Solution Box for Question 2.

Mr. Midst phoned the other families for their assistance. They examined the clue and, together, figured it out. They went home and waited for the next clue. They didn't have to wait very long.

The next morning, Mrs. Wantmore heard a rustle outside her door. Assuming it was the masked person, she rushed downstairs. She was too late. All that was left was a short clue. It said:

This one is a logic clue,
Read on to figure out what to do.
Mr. Wantmore went away,
And his garden was wrecked that day.
Claudia claimed that Bob was to blame,
Bob said Jack ruined it while playing a game.
But Jack said that he had nothing to do with it.
Exactly one person lied, so who did it?

3. All right, reader, you know what to do. Figure out who destroyed the garden. Then write the second letter of the child's name in the Solution Box for Question 3.

Mrs. Wantmore did not want to share this information with the other families, but she could not find the answer by herself. She decided to invite the other families to her house to figure it out together. Once that was accomplished, the families went home, wondering how the hunt would progress.

Late that night, the masked person put a letter in the Carings' door. Mr. Caring heard the noise and ran downstairs to find this note:

To drive 114 miles, two hours is all I need,
What unit should I use to measure my speed?

4. **Reader, think of the unit the person should use. Take the middle word of that unit and write the letters in order in each box for Question 4 in the Solution Box.**

Mr. Caring summoned the other families to his house and they proceeded to find the unit. The Wantmores and Midsts departed, leaving the Carings to ponder when this mystery would be solved.

The following day, the Midsts' youngest son, Daniel, heard footsteps. He bolted down the stairs, expecting to see a note. Instead he found a few equations and the question, "Which of these is true for the values given?"

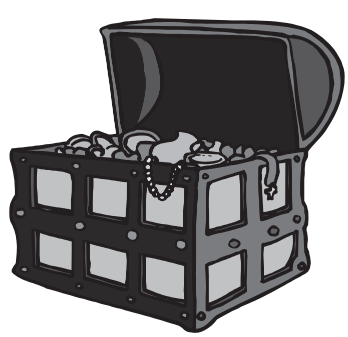
$2(e + 7) = n - 6$	$8 \div 2 + 4(e + s) = 12$	$28 - 7a = 14 + t$
for $e = 5$, $n = 23$	for $e = 1$, $s = 2$	for $a = 3$, $t = -7$

5. **Reader, it's time for you to do some more solving. Figure out which of the above is a true statement when the variables are replaced with the values given. Write the two variables in alphabetical order in each space for Question 5 in the Solution Box.**

Daniel informed his parents of the equations. His parents told the Carings and Wantmores to come and help. The families came up with their solution and went home proud of all the problems they had solved.

The next day, the Wantmores' daughter, Claudia, was awakened by a rustle outside. She found a sheet of paper under her door. It said:

This will be the final clue,
Read carefully to find out what to do.
On any rectangular prism, there are four more than eight,
So find this answer quickly—before it is too late!
What are these?



6. Well, reader, the clue is pretty clear. Find out what the masked figure is referring to. Write the first letter of that word in the Solution Box for Question 6. The Solution Box should now spell out something the three families have learned how to do. If it does, continue reading the story. If it does not, please go back and check your work on the problems you have done.

Solution Box

Letter									
Question	**1**	**2**	**3**	**4**	**4**	**4**	**5**	**5**	**6**

The three families met at the Wantmores' house and eventually obtained the answer. "What now?" questioned Mr. Wantmore. "Don't we receive a reward for our labor?" "Perhaps we'll find something tomorrow morning," suggested Mrs. Caring, "but now my family must go home. Goodbye!"

The Midsts and Wantmores waived good-bye and promised to search for a prize the next day. However, none of the families in Discovery Woods heard a sound the next morning. Each of the families searched their lawns, but found nothing. The families roamed further and further into the woods. They finally reached the middle of the woods, where all three families met to find a huge treasure chest. They worked as a group to pry it open. They peeked inside to find . . . a math book!

"All that work for nothing," mused Mr. Midst. Then he noticed the following note attached to the book:

Here is a book for everyone to use,
It will review math skills that people must not lose.
I'm sorry to say, the mystery now ends,
I hope you have all brushed up on math—and made new friends.

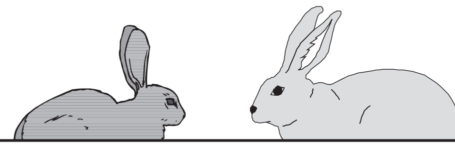
The three families sighed in unison with the exception of the Carings' son, George. George was eyeing the math book in a curious manner. "You know, that's pretty thick, even for a math book," he commented. George began leafing through the pages, and out of each one fell a crisp $1,000 bill. The Wantmores, Midsts, and Carings screamed in unison. After they had gotten over the initial shock, the families decided to split the money equally. The families thought of all the things the masked person had given them: new friends, mathematics skills, and hundreds of thousands of dollars. It was then that they noticed, on the last page of the book, a final note from the generous person who had given them so much.

I tricked you again, didn't I?

Discovering a *Rabbitly* Growing Pattern:

Then Rapidly Growing It with a Spreadsheet

Teacher's Notes



- **NCTM Standards:** Number and operations; Technology
- **CCS Standards:** Expressions and Equations; Model with mathematics. Use appropriate tools strategically.
- **Mathematical Topics:** Look for a pattern to discover the sequence of Fibonacci numbers; understand numbers expressed in scientific notation; use formulas in a spreadsheet
- **Grouping of Students:** Work individually or in small groups

Background

Most middle-school students are familiar with the Fibonacci numbers, but few are probably aware of their origins with respect to the "Rabbit Problem." In solving this problem, students use the problem-solving strategy *look for a pattern* to discover the sequence of numbers that bears Fibonacci's name.

As students analyze the activity of the rabbits during each month, you may want them to use manipulatives, such as stickers, to help them act out the situation. Use one color for males and another for females. When a pair of rabbits is old enough to breed, write a "B" on the stickers.

Examples of Fibonacci numbers occur throughout nature. They include:

- The spiral ratio (counterclockwise to clockwise) in a pine cone is often 8:5.
- The spiral ratio in a pineapple is often 13:8.
- The spiral ratio in a daisy head is often 34:21.
- The spiral ratio of sunflower seeds is often 55:34.
- In a sequence of tree branches. As you look up and across a tree, you might notice that the growth pattern of the trunk and branches is 1, 1, 2, 3, 5 . . .

For more information on the Fibonacci numbers, go to http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/rabserieswhy.html.

Once students discover the pattern of the Fibonacci sequence, the activity lesson continues with students using a computer spreadsheet to find the first 100 Fibonacci numbers. *Some prior student knowledge of spreadsheets is assumed.* The directions given in the activity lesson are based on using *Excel,* but the directions should work with other spreadsheet programs with minor modifications.

Numbers expressed in *scientific notation* is an interesting sidelight that emerges from the use of a computer spreadsheet. In many spreadsheet programs, this will occur with the 50th Fibonacci number. Since writing numbers in scientific notation is a more compact way for expressing numbers than using standard notation, many spreadsheet programs automatically use scientific notation for numbers that contain more than 11 digits. When column B is widened (in question 6i), many spreadsheet programs will continue with scientific notation beginning with the 55th Fibonacci number.

Before column B is widened, the 50th term will likely appear as 1.2586E+10. This is another way to express 1.2586×10^{10}. Both are valid forms for expressing a number in scientific notation. To write the number in *standard notation,* multiply 1.2586 by 10^{10}—or simply move the decimal point *10 places to the right* to obtain 12,586,000,000. Notice that when column B is widened (and commas are inserted), the 50th Fibonacci number is given in standard notation as 12,586,269,025. Thus, the numbers given in scientific notation are rounded. (See the Solution for question 6m for more information on such rounding.)

A number written in scientific notation must be written as a number greater than or equal to 1 and less than 10 multiplied by a power of 10. The 100th Fibonacci number is shown in scientific notation below.

a number greater than or equal to 1 and less than 10 | a power of 10

3.54225×10^{20}

3.54225 E+20

Mathematical Humor

When most people learn to count, they say 1, 2, 3, 4, and so on. Some believe that when Fibonacci learned to count, he said 1, 1, 2, 3, 5, 8, and so on. This caused his teacher to say, "This is *sum* counting!"

Solutions

1. During Month 4, rabbits A and B give birth to rabbits G and H; rabbits C and D give birth to rabbits I and J. Rabbits E and F are too young to breed.
2. 5 pairs. Rabbits A & B give birth to rabbits K & L; C & D give birth to M & N; E & F give birth to O & P. Rabbits G, H, I, and J are too young to breed.
3. 8 pairs. Rabbits A & B give birth to rabbits Q & R; C & D give birth to S & T; E & F give birth to U & V; G & H give birth to W & X; and I and J give birth to Y & Z. Rabbits K, L, M, N, O, and P are too young to breed.
4. Students should explain that each number in the sequence, after the first two numbers, is found by finding the sum of the previous two.

5a. Month 7: 13 pairs; Month 8: 21 pairs; Month 9: 34 pairs; Month 10: 55 pairs; Month 11: 89 pairs; Month 12: 144 pairs.

b. 144 pairs of rabbits

6a–d. Check students' spreadsheets.

e. The term numbers from 1 through 100 appear in column A.

f. 2; Add the two previous Fibonacci numbers; Cell B5 will hold the third Fibonacci number. Since we need to add the two previous Fibonacci numbers, the following formula should be used: =B3+B4.

g. =B4+B5

h. Answers will vary. Students may respond that at some point the Fibonacci numbers will appear in scientific notation. With many spreadsheet programs, this occurs with the 50th Fibonacci number.

i–j. Students should obtain the spreadsheet at the right and on page 183.

k. 12,586,269,025

l. 354,224,848,179,262,000,000 (This number is read 354 quintillion, 224 quadrillion, 848 trillion, 179 billion, 262 million.)

m. Answers will vary. Beginning with the 74th Fibonacci number, students may notice that "rounded" values are being used. For example, the 72nd and 73rd Fibonacci numbers end with 4 and 3, respectively, but the 74th Fibonacci number ends with 0. So, the 74th term is not the (exact) sum of the two previous terms. This continues as you scroll down, with more and more ending place-value positions containing 0s. Evidently, when a spreadsheet renders numbers in scientific notation, it maintains only a limited number of *significant digits.* (Note that although the 100th Fibonacci number, 3.54225E+20, contains six significant digits when rendered in scientific notation, the spreadsheet actually stores about 15 significant digits—as evidenced by the rendering of the number in standard notation.)

The First 100 Fibonacci Numbers	
Term	**Fibonacci Number**
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
9	34
10	55
11	89
12	144
13	233
14	377
15	610
16	987
17	1,597
18	2,584
19	4,181
20	6,765
21	10,946
22	17,711
23	28,657
24	46,368
25	75,025
26	121,393
27	196,418
28	317,811
29	514,229
30	832,040
31	1,346,269
32	2,178,309
33	3,524,578
34	5,702,887
35	9,227,465
36	14,930,352
37	24,157,817
38	39,088,169
39	63,245,986

40	102,334,155
41	165,580,141
42	267,914,296
43	433,494,437
44	701,408,733
45	1,134,903,170
46	1,836,311,903
47	2,971,215,073
48	4,807,526,976
49	7,778,742,049
50	12,586,269,025
51	20,365,011,074
52	32,951,280,099
53	53,316,291,173
54	86,267,571,272
55	139,583,862,445
56	225,851,433,717
57	365,435,296,162
58	591,286,729,879
59	956,722,026,041
60	1,548,008,755,920
61	2,504,730,781,961
62	4,052,739,537,881
63	6,557,470,319,842
64	10,610,209,857,723
65	17,167,680,177,565
66	27,777,890,035,288
67	44,945,570,212,853
68	72,723,460,248,141
69	117,669,030,460,994
70	190,392,490,709,135
71	308,061,521,170,129
72	498,454,011,879,264
73	806,515,533,049,393
74	1,304,969,544,928,660
75	2,111,485,077,978,050
76	3,416,454,622,906,710
77	5,527,939,700,884,760
78	8,944,394,323,791,460
79	14,472,334,024,676,200
80	23,416,728,348,467,700

81	37,889,062,373,143,900
82	61,305,790,721,611,600
83	99,194,853,094,755,500
84	160,500,643,816,367,000
85	259,695,496,911,123,000
86	420,196,140,727,490,000
87	679,891,637,638,612,000
88	1,100,087,778,366,100,000
89	1,779,979,416,004,710,000
90	2,880,067,194,370,820,000
91	4,660,046,610,375,530,000
92	7,540,113,804,746,350,000
93	12,200,160,415,121,900,000
94	19,740,274,219,868,200,000
95	31,940,434,634,990,100,000
96	51,680,708,854,858,300,000
97	83,621,143,489,848,400,000
98	135,301,852,344,707,000,000
99	218,922,995,834,555,000,000
100	354,224,848,179,262,000,000

Extension

The Extension brings out a surprising connection between the Fibonacci numbers and the golden ratio. Students first compute some ratios by hand and then use their spreadsheet to show the ratios through the first 100 Fibonacci numbers. The ratios of consecutive Fibonacci numbers approach the golden ratio—an irrational number that is equal to 1.6180339887498948482 . . . (often denoted by Φ). An irrational number is a number that has an infinite, non-repeating decimal expansion. Examples of irrational numbers include π and $\sqrt{2}$.

In using their spreadsheet to compute the ratios, students should enter the following formula into cell C4: **=B4/B3.** By using the Fill Down function, students should obtain **=B5/B4** in cell C5, **=B6/B5** in cell C6, and so on.

An interesting sidelight relates to what happens when you apply the summing aspect of the Fibonacci sequence to a growing pattern other than the Fibonacci numbers. Suppose you begin a sequence with, say, 39 and 43. The next terms in a summing sequence would be 82, 125, 207, 332, and so on. The ratios of consecutive numbers in this sequence will likewise approach the golden ratio.

Solutions

1a. $\frac{1}{1} = 1$ $\quad \frac{2}{1} = 2$ $\quad \frac{3}{2} = 1.5$

$\frac{5}{3} \approx 1.67$ $\quad \frac{8}{5} = 1.6$ $\quad \frac{13}{8} \approx 1.63$

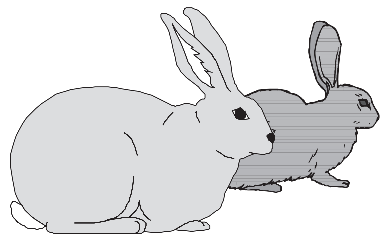
$\frac{21}{13} \approx 1.62$ $\quad \frac{34}{21} \approx 1.62$ $\quad \frac{55}{34} \approx 1.62$

b. 1.62; Yes, the golden ratio

2. 1.618033989

Name____ Date ___________________

Discovering a *Rabbitly* Growing Pattern



Materials: computer; spreadsheet software (such as *Excel*)

One of the earliest European writers of mathematics was Leonardo of Pisa (1175?–1240?), known today by the name Fibonacci (son of Bonaccio). His masterpiece, *Liber abbaci,* or *Book of Calculation,* was published in 1202. The word *abbaci* comes from the Latin *abacus.* Fibonacci gained much of his knowledge in mathematics during his many journeys throughout the Islamic world. Through his work, Fibonacci introduced the Hindu-Arabic numerals (0–9) and methods of arithmetic into European culture.

The most famous problem in Fibonacci's book is the "Rabbit Problem" (restated below). It deals with how fast rabbits can breed under "ideal conditions."

> A pair of newly born rabbits (one male and one female) is placed in an area surrounded on all sides by a wall. We assume that rabbits are able to mate *after* 1 month, and we assume that each month (beginning with Month 2 of a rabbit's life) each pair of rabbits gives birth to another pair (one male and one female). How many pairs of rabbits will there be at the beginning of Month 12?

In the table on the following page, a different letter designates each rabbit. The table shows how many pairs there are at the beginning of each month and the activity that takes place during the month.

- At the *beginning* of Month 1, there is 1 pair of rabbits: A & B. No rabbits are born during this month since the rabbits are not more than 1 month old.
- At the *beginning* of Month 2, there is still 1 pair of rabbits: A & B. However, *during* the month, Rabbits A & B give birth to Rabbits C & D. (Rabbits listed in bold type in the table are breeding during that month.)
- At the *beginning* of Month 3, there are 2 pairs of rabbits: A & B and C & D. *During* Month 3, Rabbits A and B give birth to Rabbits E and F. Rabbits C and D do not give birth during this month because they are only 1 month old. (Rabbits listed in the third row of the table do not breed during that month.)

Month 1	Month 2	Month 3	Month 4	Month 5
Pairs of Rabbits at the *Beginning* of Each Month				
1 pair	1 pair	2 pairs	3 pairs	_____ pairs
Activity *During* the Month				
A, B	**A B** ↓ C D	**A B** ↓ E F C, D	**A B** ↓ G H **C D** ↓ I J E, F	**A B** **C D** **E F**

Questions

1. Explain what is happening during Month 4.

____

____

2. How many pairs of rabbits are there at the beginning of Month 5? ___________
Complete the above table to show what happens during Month 5.

3. How many pairs of rabbits are there at the beginning of Month 6? ___________
Complete the table below for Month 6.

Month 6
______ **pairs at beginning of month**
A B   **C D**   **E F**   **G H**   **I J**

4. Look at how the number of pairs of rabbits at the beginning of each month grows. Put on your detective hat to find a pattern. Describe your pattern below.

5a. In the table below, use your pattern to extend the results to show how many rabbits there are at the beginning of each of Months 7–12.

Month 7	Month 8	Month 9	Month 10	Month 11	Month 12
_____ pairs	_____ pairs	_____ pairs	_____ pairs	_____ pairs	_____ pairs

b. What is the solution to Fibonacci's "Rabbit Problem"? ______________________________

The sequence of numbers you have discovered, 1, 1, 2, 3, 5, 8, 13, 21 . . . is known as the *Fibonacci numbers.* We're now going to examine a way to quickly list many more terms of this *infinite* sequence. How quickly do you suppose you could name the 50th term of this sequence? How about the 100th term? About how large a number do you suppose the 100th term is? One way to help us answer these questions is to use a computer *spreadsheet.*

6. Follow these instructions to create a spreadsheet to display the first 100 Fibonacci numbers.

a. In cell A1, enter the words *The First 100 Fibonacci Numbers.*

b. In cell A2, enter the word *Term.*

c. In cell B2, enter the words *Fibonacci Number.*

d. You may want to boldface each of the above entries. You may also want to center the word *Term* in its cell and *Fibonacci Number* in its cell. To do that, highlight the cells and click on B in the toolbar (for boldface) and click on the toolbar tool to center the text.

e. In column A, we will list the Term numbers from 1–100. We could enter each of the whole numbers 1–100 individually, but it's much quicker using a formula. Enter **1** in cell A3 (to denote term #1). In cell A4, enter **=A3+1.** This will add 1 to the value in cell A3 (1) and place the sum (2) in cell A4 (to denote term #2). In cell A5, enter **=A4+1.** This will add 1 to the value in cell A4 (2) and place the sum (3) in cell A5 (to denote term #3).

We could keep on entering formulas, but this would amount to more work than simply entering the numbers 1–100. Instead, we can use the Fill Down function to speed up our work. Use your mouse to highlight cell A5. With your finger firmly pressing down on the clicker, scroll down column A about 100 rows. You have now highlighted about 100 cells in column A. Then go to the Edit menu, select Fill, and then Down. Then release the clicker.

What do you now have in the cells in column A? ___________________________________

(The Fill Down function progressively increases by 1, from cell to cell, any cell locations identified in a formula. So, cell A6 will now contain the formula =A**5**+1, cell A7 will contain =A**6**+1, and so on. If you click on any cell in column A, you can see the formula contained in the cell by looking at window of the tool bar.)

f. We are now going to enter the Fibonacci numbers into column B. In cell B3, enter 1 (the first Fibonacci number). In cell B4, enter 1 (the second Fibonacci number). What is the third Fibonacci number, and how is it obtained?

________ ; ____

We could manually enter each Fibonacci number in column B, but that would be a lot of work. Instead we should use a formula.

Explain how you can determine the formula for cell B5.

____

____

g. In cell B5, enter **=B3+B4.** What formula should we enter in cell B6? ______________

As before, instead of manually entering a formula in each cell, we can use the Fill Down function. Highlight cell B5, and fill down about 100 rows. Click on the cells in column B and check the window in the toolbar to see how the formula evolves from row to row.

h. Slowly scroll down to the 100th Fibonacci number. Do you notice a change in the format of the numbers? What do you call numbers written in this format?

____

i. Since many of the numbers do not fit in column B, we can widen the column. Place your cursor on the right-hand vertical bar of the B column header of the spreadsheet. While pressing and holding the clicker, drag the vertical bar to the right to widen the column.

j. As you scroll down, notice that some of the numbers are still in scientific notation. To change the format of the numbers, highlight the entire B column by clicking on the cell B column header cell. Then go to Format menu, then Cells. When you release the clicker, the Format Cells menu will appear. Click on "Number," select "0 Decimal places," and check "Use 1000 Separator (,)." Then click OK. The Fibonacci numbers should now all appear in standard notation (with commas).

k. What is the 50th Fibonacci number? ____________________________________

l. What is the 100th Fibonacci number? ____________________________________

m. Closely examine your spreadsheet. Does it appear that the computer made "errors" in calculating some of the Fibonacci numbers? Explain.

____

____

____

Extension

1a. We are now going to discover a very special relationship among the Fibonacci numbers. Divide each Fibonacci number listed below by its preceding Fibonacci number. Round each result to the nearest hundredth.

Fibonacci Numbers

1	1	2	3	5	8	13	21	34	55	. . .

$\frac{1}{1}$ = ______ $\frac{2}{1}$ = ______ $\frac{3}{2}$ = ______ $\frac{5}{3}$ ≈ ______ $\frac{8}{5}$ = ______

$\frac{13}{8}$ ≈ ______ $\frac{21}{13}$ ≈ ______ $\frac{34}{21}$ ≈ ______ $\frac{55}{34}$ ≈ ______

b. Examine the above ratios of consecutive Fibonacci numbers. What number do the ratios appear to be approaching? Does this number have a special name?

______________ ; ____

2. Refer to the spreadsheet that you created. In column C, write a formula that will divide each Fibonacci number by its preceding Fibonacci number, beginning with the second Fibonacci number. What value does this ratio appear to be approaching? Give your answer to the billionths place (9 decimal places).

____

Uncovering Humorous Mathematical Blunders

Teacher's Notes

- **NCTM Standards:** Number and Operations; Geometry; Algebra; Data Analysis and Probability
- **CCS Standards:** Operations and Algebraic Thinking; Measurement and Data; Attend to precision.
- **Mathematical Topics:** Estimation; finding the percent increase; fraction/percent conversions; addition of fractions; ordering numbers; sum of the degree measures of the central angles of a circle; combining like terms; median; probability of compound events
- **Grouping of Students:** Work in pairs or small groups or individually

Background

Unfortunately, despite years of study and life experience in an environment immersed in quantitative data, many educated adults remain functionally innumerate.

—From *The Case for Quantitative Literacy* by Lynn Arthur Steen (2000)

For many years the author has been collecting mathematical blunders such as those in this activity lesson. The author has used such material in his classes, mathematics presentations at conferences, and after-dinner appearances to draw attention to mathematics illiteracy in the real world—and to the ridiculous conclusions that result. Essentially, the use of this material illustrates the fact that the kinds of errors students make in the classroom are often repeated by the same people—as adults—in the real world of work. If students observe how foolish some people appear in the real world due to their mathematical illiteracy, perhaps this will provide some incentive for them to improve their performance as students in the classroom.

Mathematical Humor

Sign at a grocery store: "Stock up and Save. Limit 1."

Sign at grocery store: "Open 24 hours on Labor Day: 9:00 A.M.–9:00 P.M."

Sign on road leading to tunnel: "When this sign is under water, the tunnel is impassable."

Baseball player: "We just beat the Braves four games out of three."

Youngster at restaurant: "Do you give free refills?"
Waitress: "Yes, but you have to pay for them."

Sign in a subway car: "Illiterate? Write today for free help."

TV football announcer discussing cleat sizes during a nationally televised game:
"Which is more, one-half inch or five-eighths inch?"

Solutions

1a. The age of the dinosaur bones should be given as an estimate, not as an exact amount.

b. The dinosaur bones are about 70 million years old.

2a. The new 2-liter size contains 100% more than the 1-liter size.

b. When some people see a 1 and a 2, they think "$\frac{1}{2}$," or 50%. Others think twice the size means 50%.

3a. Turning something around 360° gets you back to where you started.

b. The player meant to say 180°. This would result in the opposite effect of the current situation.

4. Since $\frac{3}{4} + \frac{1}{3} = \frac{13}{12}$, a value greater than 1, this is an impossible situation. The whole cannot be greater than 100%.

5a. The sign attempts to find a sum of unrelated data involving different units of measure.

b. Students may solve the problem in any of these ways:

49 mm + 8 cm + 20 m = 0.049 m + 0.08 m + 20 m, or 20.129 m

49 mm + 8 cm + 20 m = 4.9 cm + 8 cm + 2,000 cm, or 2,012.9 cm

49 mm + 8 cm + 20 m = 49 mm + 80 mm + 20,000 mm, or 20,129 mm

c. Convert all measurements so that they are expressed in the same unit.

6a. The "interval" between three-quarters and 75% is 0. So the baseball player was redundant in saying that he was about three-quarters to 75% ready to play. (Comedian David Letterman once said: "There is a new survey—apparently three out of every four people make up 75% of the population.")

b. $\frac{1}{2}$ and 50% name the same amount. Also, $\frac{1}{3}$ and $33\frac{1}{3}$% name the same amount.

7a. By definition, 50% of students are above the median and 50% are below the median. So it is impossible for every child in the district to ever be above the district median.

b. Yes. Since, by definition, 50% of the students are above the district median test score, the district is halfway toward the 100% goal of School Board Member B. Unfortunately, the district will remain being halfway toward that goal.

8a. The answer choice "Not here" could potentially be any number—including those that are larger than the largest number in the list, 6.17. The question, "Which is the largest number?" considered together with the answer choice "Not here" might suggest that the question is getting at the fact that there is no largest number. The "correct" answer given by the answer key for this most confusing test item was 6.17.

b. Answers will vary. Answer choice E could be changed to a specific value. Another option is to change the question to "Which is the largest number listed below?"

9a. 0.50

b. 0.50

c. P(no rain on both days) = $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ (or 25%)

d. P(rain on at least on day) = 1 − P(no rain on both days) = $1 - \frac{1}{4} = \frac{3}{4}$ (or 75%)

e. The situation involves compound probabilities that need to be multiplied—along with consideration given to the various possible cases (rain on one day but not on the other, and rain on both days). The weather forecaster added the probabilities to conclude that rain was certain over the weekend. This, of course, is incorrect because it was certainly possible for no rain to occur on either day.

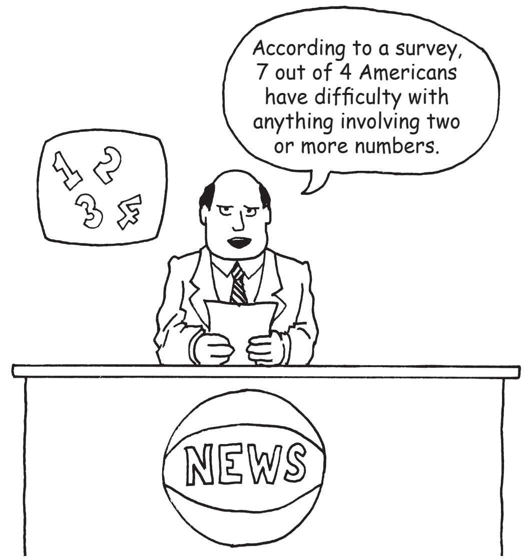
Extension

Ask students to search for mathematical blunders that may appear in newspapers, magazines, store advertisements, newscasts, and more. Have them present the blunders to the class. Have students suggest ways to fix the blunders.

Name____ Date ____________________

UNCOVERING HUMOROUS MATHEMATICAL BLUNDERS

The following mathematical blunders actually took place in the real world. Some are rather humorous in nature. The names of the people and companies guilty of the blunders have been omitted to "protect their identities."



1. **Museum Docent:** "These dinosaur bones are exactly 70,000,006 years old."
   Museum Visitor: "How do you know that they are *exactly* that old?"
   Museum Docent: "Well, 6 years ago when I got this job, they told me they were 70,000,000 years old. So, 6 + 70,000,000 = 70,000,006."

   a. Uncover the docent's mathematical blunder.

   ____

   b. What should the museum docent have said?

   ____

2. **Try our new 2-liter size.**
   It contains 50% more
   than the 1-liter size.

   a. Uncover the mathematical blunder in this ad.

   ____

   b. Why do you suppose this is such a common error?

   ____

3. **Basketball Player:** "Now that I'm joining the Mavericks, we're going to turn around the program 360 degrees."

a. Uncover the basketball player's mathematical blunder.

____

b. What do you think the basketball player meant to say?

____

4.

Job Opportunities
Mathematics Research Assistant
This is a $\frac{3}{4}$ research and $\frac{1}{3}$ teaching position.

Uncover the mathematical blunder in this ad.

____

5.

SNOWMASS VILLAGE	
POPULATION	**1,018**
ELEVATION	**8,388**
ESTABLISHED	**1967**
TOTAL	**11,373**

a. This sign was probably created in jest. What mathematical "blunder" is being illustrated here?

____

____

____

b. Find the sum of these measurements: 49 mm + 8 cm + 20 m. ________

c. What did you do in order to find the sum?

____

6. **Baseball Player:** "With my leg almost healed, I am about three-quarters to 75 percent ready to play."

a. Uncover the baseball player's mathematical blunder.

____

b. Circle all pairs of numbers below that name the same amount.

$\frac{1}{2}$ and 50% $\frac{1}{3}$ and $33\frac{1}{3}$% $\frac{1}{8}$ and 8% $\frac{2}{5}$ and 25%

7. **School Board Member A** (commenting on local test results): "Within three years, I expect every child in the district to be above the district median test score."

   School Board Member B: "Well, I guess we're already halfway toward your goal."

   a. Uncover the mathematical blunder committed by School Board Member A.

   ____

   b. Was the comment made by School Board Member B accurate? Explain.

   ____

8. Item on a multiple-choice test:

Which is the largest number?
A. 6.097
B. 6.0971
C. 6.096711
D. 6.17
E. Not here

   a. The answer given in the answer key for this item was D, 6.17. Uncover the mathematical blunder committed by the writer of this test item.

   ____

   ____

   ____

   b. How would you rewrite this test item? Include the answer for your test item.

   ____

9. **Weather Forecaster:** "There's a 50% chance for rain on Saturday, and a 50% chance for rain on Sunday. Well, I guess we're certain to have rain this weekend."

   a. What is the probability for no rain on Saturday? __________

   b. What is the probability for no rain on Sunday? __________

   c. What is the probability for no rain on both days? __________
   Hint: P(no rain on both days) = P(no rain on Sat.) × P(no rain on Sun.)

   d. What is the probability that there will be rain on *at least one* of the two days? __________
   Hint: P(rain on at least one day) = 1 – P(no rain on both days)

   e. Describe the weather forecaster's mathematical blunder.

   ____

   ____

   ____

I know What You're Thinking:
It's Not Magic, It's Algebra!
Teacher's Notes

- **NCTM Standards:** Number and Operations; Algebra
- **CCS Standards:** Operations and Algebraic Thinking; Look for and make use of structure.
- **Mathematical Topics:** Perform basic computations; write algebraic expressions based on given conditions
- **Grouping of Students:** Work independently or in pairs

Background

This activity lesson requires students to carefully follow a series of directions that involve a review of some basic number concepts. The ultimate goal is not to have students come away thinking that they have learned a cute magic trick. Rather, the goal is for students to want to know *why* the teacher "knows what he or she is thinking." Students who are challenged to use the power of algebra to justify why the "trick" works (see Extension) should conclude that this is not a magic trick at all. Rather, these students should conclude that the reason why the "trick" works is elegantly explained through the use of algebraic generalizations.

The NCTM Process Standard, Reasoning and Proof highlights the importance of *justification* in mathematics (NCTM, 2000, p. 262). According to this standard, middle-school students should—

- examine patterns and structures to detect regularities;
- formulate generalizations and conjectures about observed regularities;
- evaluate conjectures; and
- construct and evaluate mathematical arguments.

Mathematical Humor

Teacher: "Define algebra."

Student: "Algebra is the intense study of the final three letters of the alphabet."

Note: A much better definition is, "Algebra is generalized arithmetic."

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

Old math teachers never die . . . they just lose some of their functions.

Solution

After students go through steps 1–11, ask them, “Are you thinking of an eagle, emu, egret, or Egyptian vulture from Denmark?” Otherwise, it’s possible that they might be thinking of an ostrich, owl, osprey, or oriole from the Dominican Republic—or a jay from Djibouti! Another possibility is an umbrella bird from Dubai. Other solutions are possible.

Extension

Challenge students to use *algebra* to prove *why* the “trick” works. The steps in the algebraic justification below provide such an analysis.

1. Let x = your number.
2. Subtracting 5 gives us $x - 5$.
3. Multiplying by 3 gives us $3(x - 5) = 3x - 15$.
4. Squaring the number gives us
   $(3x - 15)(3x - 15) = 9x^2 - 90x + 225 = 9(x^2 - 10x + 25)$.
5. The result in step 4 above is a multiple of 9, since it is a product of 9 and some other quantity. The sum of the digits of a multiple of 9 is always a multiple of 9. (For example, consider 675. This is a multiple of 9, since 9 × 75 = 675. The sum of the digits in 675 is 6 + 7 + 5 = 18, a multiple of 9. The sum of the digits in 18 is 1 + 8 = 9, a multiple of 9.)
   So as step 5 is carried out until you get only one digit, ultimately the end result will always be 9, unless $x = 5$. When $x = 5$, $x^2 - 10x + 25 = 52 - 10(5) + 25 = 0$. So when $x = 5$, the end result will be 0.
6. The result in this step will always be 5, as explained below.
   - If the result in step 5 was 0, you do the computation 0 + 5 = 5.
   - If the result in step 5 was 9, you do the computation 9 – 4 = 5.
7. Multiplying 5 by 2 yields 10.
8. Subtracting 6 yields 4. So no matter what number you pick in step 1, your end result will always be 4.
9. 4 maps to the letter D.
10. Most people select Denmark. The Dominican Republic or Djibouti are also possible responses.
11. The second letter in Denmark is E. Most people pick eagle. Some people pick emu, egret, or Egyptian vulture. The second letter in Dominican Republic is O, with ostrich, owl, osprey, or oriole selected. A few students might be thinking of a jay from Djibouti.

Name____ Date ___________________

I Know What You're Thinking:
It's Not Magic, It's Algebra!

1. Pick an integer from -9 through 9.

2. Subtract 5.

3. Multiply by 3.

4. Square the number.

5. Add the digits in your result until you get only one digit.
   For example: $64 \rightarrow 6 + 4 = 10$; $10 \rightarrow 1 + 0 = 1$.

6. If your result in Step 5 is less than 5, add 5. Otherwise, subtract 4.

7. Multiply by 2.

8. Subtract 6.

9. Map the digit to a letter in the alphabet as follows: $1 \rightarrow$ A, $2 \rightarrow$ B, $3 \rightarrow$ C, etc.

10. Pick a name of a country that begins with that letter.

11. Take the second letter in the country name and think of a bird whose name begins with that letter.

    What bird did you choose? ____________________

    What country did you choose? ____________________

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